Electric Circuit Theory and the Operational Calculus

By JOHN R. CARSON

NOTE: This is the first of three installments by Mr. Carson which will embody material given by him in a course of lectures at the Moore School of Electrical Engineering, University of Pennsylvania, May, 1925. No effort has been spared by the author to make his treatment clear and as simple as the subject matter will permit. The method of presentation is distinctively pedagogic. To electrical engineers and to engineering instructors, this exposition of the fundamentals of electric circuit theory and the operational calculus should be of great value.—EDITOR.

FOREWORD

THE following pages embody, substantially as delivered, a course of fifteen lectures given during the Spring of 1925 at the Moore School of Electrical Engineering of the University of Pennsylvania.

After a brief introduction to the subject of electric circuit theory, the first chapters are devoted to a systematic and fairly complete exposition and critique of the Heaviside Operational Calculus, a remarkably direct and powerful method for the solution of the differential equations of electric circuit theory.

The name of Oliver Heaviside is known to engineers the world over: his operational calculus, however, is known to, and employed by, only a relatively few specialists, and this notwithstanding its remarkable properties and wide applicability not only to electric circuit theory but also to the differential equations of mathematical physics. In the writer's opinion this neglect is due less to the intrinsic difficulties of the subject than to unfortunate obscurities in Heaviside's own exposition. In the present work the *operational calculus* is made to depend on an integral equation from which the Heaviside Rules and Formulas are simply but rigorously deducible. It is the hope of the writer that this mode of approach and exposition will be of service in securing a wider use of the operational calculus by engineers and physicists, and a fuller and more just appreciation of its unique advantages.

The second part of the present work deals with advanced problems of electric circuit theory, and in particular with the theory of the propagation of current and voltage in electrical transmission systems. It is hoped that this part will be of interest to electrical engineers generally because, while only a few of the results are original with the present work, most of the transmission theory dealt with is to be found only in scattered memoirs, and there accompanied by formidable mathematical difficulties.

While the method of solution employed in the second part is largely that of the operational calculus, I have not hesitated to employ developments and extensions not to be found in Heaviside. For example, the formulation of the problem as a Poisson integral equation is an original development which has proved quite useful in the actual numerical solution of complicated problems. The same may be said of the Chapter on Variable Electric Circuit Theory.

In view of its two-fold aspect this work may therefore be regarded either as an exposition and development of the operational calculus with applications to electric circuit theory, or as a contribution to advanced electric circuit theory, depending on whether the reader's viewpoint is that of the mathematician or the engineer.

I have not attempted in the text to give adequate reference to the literature of the subject, now fairly extensive. In an appendix, however, there is furnished a list of original papers and memoirs, for which, however, no claim to completeness is made.

CHAPTER I

The Fundamentals of Electric Circuit Theory

While a knowledge, on the reader's part, of the elements of electric circuit theory will be assumed, it seems well to start with a brief review of the fundamental physical principles of circuit theory, the mode of formulating the equations, and some general theorems which will prove useful subsequently.

First, the *circuit elements* are resistances, inductances, and condensers. The network is a *connected* system of circuits or branches each of which may include resistance, inductance and capacitance elements together with mutual inductance, and mutual branches.

The equations of circuit theory may be established in a number of different ways. For example, they may be based on Maxwell's dynamical theory. In accordance with this method, the network forms a dynamic system in which the currents play the role of velocities. If we therefore set up the expressions for the kinetic energy, potential energy and dissipation, the network equations are deducible from general dynamic equations.

The simplest, and for our purposes, a quite satisfactory basis for the equations of circuit theory are found in Kirchhoff's Laws. These laws state that

I. The total impressed force taken around any closed loop or circuit in the network is equal to the potential drop due to (a) resistance, (b) inductive reaction and (c) capacitive reactance.

2. The sum of the currents entering any branch point in the network is always zero.

Let us now apply these laws to an elementary circuit in order to deduce the physical significance of the circuit elements.

Consider an elementary circuit consisting of a resistance element R, an inductance element L and a capacity element C in series, and let an electromotive force E be applied to this circuit. If I denote the current in the circuit, the resistance drop is RI, the inductance drop is LdI/dt and the drop across the condenser is Q/C where Q is the charge on the condenser. It is evident that Q and I are related by the equation $I=dQ/dt$ or $Q=\int Idt$. Now apply Kirchhoff's law relating to the drop around the circuit: it gives the equation

$$RI+LdI/dt+Q/C=E.$$

Multiply both sides by I: we get

$$RI^2+\frac{d}{dt}\frac{1}{2}LI^2+\frac{d}{dt}\frac{Q^2}{2C}=EI.$$

The right hand side is clearly the rate at which the impressed force is delivering energy to the circuit, while the left hand side is the rate at which energy is being absorbed by the circuit. The first term RI^2 is the rate at which electrical energy is being converted into heat. Hence the resistance element may be defined as a device for converting electrical energy into heat. The second term $\frac{d}{dt}\frac{1}{2}LI^2$ is the rate of increase of the magnetic energy. Hence the inductance element is a device for storing energy in the magnetic field. The third term $\frac{d}{dt}Q^2/2C$ is the rate of increase of the electric energy. Hence the condenser is a device for storing energy in the electric field.

In the foregoing we have isolated and idealized the circuit elements. Actually, of course, every circuit element dissipates some energy in the form of heat and stores some energy in the magnetic field and some in the electric field. The analysis of the actual circuit element, however, into three ideal components is quite convenient and useful, and should lead to no misconception if properly interpreted.

Now consider the general form of network possessing n independent meshes or circuits. Let us number these from 1 to n, and let the corresponding mesh currents be denoted by $I_1, I_2 \ldots . I_n$. Let electromotive forces $E_1, E_2 \ldots . E_n$ be applied to the n meshes or circuits respectively. Let L_{jj}, R_{jj}, C_{jj} denote the total inductance,

resistance and capacity in series in mesh j and let L_{jk}, R_{jk}, C_{jk} denote the corresponding mutual elements between circuit j and k. Now write down Kirchhoff's equation for any circuit or mesh, say mesh 1; it is

$$\left(L_{11}\frac{d}{dt}+R_{11}+\frac{1}{C_{11}}\int dt\right)I_1+\left(L_{12}\frac{d}{dt}+R_{12}+\frac{1}{C_{12}}\int dt\right)I_2+ \\ \ldots.+\left(L_{1n}\frac{d}{dt}+R_{1n}+\frac{1}{C_{1n}}\int dt\right)I_n=E_1$$

Corresponding equations hold for each and every one of the n meshes of the network. Writing them all down, we have the system of equations

$$\begin{array}{c} \left(L_{11}\frac{d}{dt}+R_{11}+\frac{1}{C_{11}}\int dt\right)I_1+\ldots+\left(L_{1n}\frac{d}{dt}+R_{1n}+\frac{1}{C_{1n}}\int dt\right)I_n=E_1 \\ \text{-------------------------------------} \\ \left(L_{n1}\frac{d}{dt}+R_{n1}+\frac{1}{C_{n1}}\int dt\right)I_1+\ldots+\left(L_{nn}\frac{d}{dt}+R_{nn}+\frac{1}{C_{nn}}\int dt\right)I_n=E_n \end{array} \quad (1)$$

The system of simultaneous differential equations (1) constitute the canonical equations of electric circuit theory. The interpretation and solution of these equations constitute the subject of Electric Circuit Theory, and it is in connection with their solution that we find the most direct and logical introduction to the Operational Calculus.

As an example of the appropriate mode of setting up the circuit equations, consider the two mesh network shown in sketch 1. Writing down Kirschhoff's Law for meshes 1 and 2, respectively, we have

$$\left(L_1\frac{d}{dt}+R_1+\frac{1}{C_1}\int dt\right)I_1+M\frac{d}{dt}I_2=E_1$$

$$+M\frac{d}{dt}I_1+\left(L_2\frac{d}{dt}+R_2+\frac{1}{C_2}\int dt\right)I_2=E_2$$

In this case the self and mutual coefficients are given by

$$\begin{array}{lll} L_{11}=L_1 & L_{22}=L_2 & L_{12}=L_{21}=+M \\ C_{11}=C_1 & C_{22}=C_2 & C_{12}=C_{21}=o \\ R_{11}=R_1 & R_{22}=R_2 & R_{12}=R_{21}=o \end{array}$$

The conventions adopted for the positive directions of currents and voltages are indicated by the arrows. The sign of the mutual inductance M will depend on the relative mode of winding of the two coils.

Now write down Kirchhoff's Law, or the circuital equation for the network of sketch 2. They are

$$\left\{(L_1+L_3)\frac{d}{dt}+(R_1+R_3)+\left(\frac{1}{C_1}+\frac{1}{C_3}\right)\int dt\right\}I_1$$

$$-\left(L_3\frac{d}{dt}+R_3+\frac{1}{C_3}\int dt\right)I_2=E_1,$$

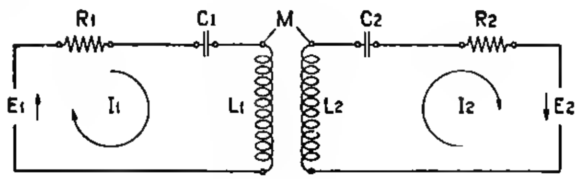
$$-\left(L_3\frac{d}{dt}+R_3+\frac{1}{C_3}\int dt\right)I_1$$

$$+\left\{(L_2+L_3)\frac{d}{dt}+(R_2+R_3)+\left(\frac{1}{C_2}+\frac{1}{C_3}\right)\int dt\right\}I_2=E_2.$$

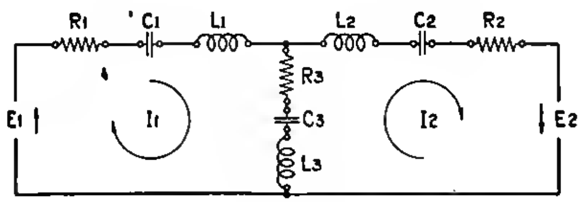
Comparison with equations (1) shows that

$$L_{11}=L_1+L_3 \qquad L_{22}=L_2+L_3 \qquad L_{12}=L_{21}=-L_3$$
$$R_{11}=R_1+R_3 \qquad R_{22}=R_2+R_3 \qquad R_{12}=R_{21}=-R_3$$
$$\frac{1}{C_{11}}=\frac{1}{C_1}+\frac{1}{C_3} \qquad \frac{1}{C_{22}}=\frac{1}{C_2}+\frac{1}{C_3} \qquad \frac{1}{C_{12}}=\frac{1}{C_{21}}=-\frac{1}{C_3}.$$

It should be observed that the signs of the mutual coefficients R_{12}, L_{12}, C_{12} are a matter of convention. For example if the conventional directions of I_2 and E_2 are reversed, the signs of the mutual coefficients are reversed.



Sketch 1



Sketch 2

The system of equations (1) possesses two important properties which are largely responsible for the relative simplicity of classical electric circuit theory. First, the equations are linear in both currents and applied electromotive forces. Secondly, the coefficients L_{jk}, R_{jk}, C_{jk} are constants. Important electrotechnical problems exist,

in which these properties no longer obtain. The solution, however, for the restricted system of linear equations with constant coefficients is fundamental and its solution can be extended to important problems involving non-linear relations and variable coefficients. These extensions will be taken up briefly in a later chapter.

Another important property is the reciprocal relation among the coefficients; that is $L_{jk}=L_{kj}$: $R_{jk}=R_{kj}$, and $C_{jk}=C_{kj}$. It is easily shown that these reciprocal relations mean that there are no concealed sources or sinks of energy. Again important cases exist where the reciprocal relations do not hold. Such exceptions, however, while of physical interest do not affect the mathematical methods of solution, to which the reciprocal relation is not essential.

Returning to equation (1) we shall now derive the *equation of activity.* Multiply the first equation by I_1, the second by I_2, etc. and add: we get

$$\frac{d}{dt}\sum\sum\frac{1}{2}L_{jk}I_jI_k+\frac{d}{dt}\sum\sum\frac{1}{2}\frac{1}{C_{jk}}Q_jQ_k+\sum\sum R_{jk}I_jI_k=\sum E_jI_j. \tag{2}$$

The right hand side is the rate at which the applied forces are supplying energy to the network. The first term on the left is the rate of increase of the magnetic energy

$$\frac{1}{2}\sum\sum L_{jk}I_jI_k,$$

while the second term is the rate of increase of the electric energy

$$\frac{1}{2}\sum\sum\frac{1}{C_{jk}}Q_jQ_k.$$

The last term, $\sum\sum R_{jk}I_jI_k$, is the rate at which electromagnetic energy is being converted into heat in the network. Consequently in the electrical network, the magnetic energy is a homogeneous quadratic function of the currents, the electric energy is a homogeneous quadratic function of the charges, and the rate of dissipation is a homogeneous quadratic function of the currents. In Maxwell's dynamical theory of electrical networks, these relations were written down at the start and the circuit equations then derived by an application of Lagrange's dynamic equations to the homogeneous quadratic functions.

Returning to equations (1), we observe that, due to the presence of the integral sign, they are integro-differential equations. They are,

however, at once reducible to differential equations by the substitution $I=dQ/dt$, whence they become

$$
\begin{array}{c}
\left(L_{11}\frac{d^2}{dt^2}+R_{11}\frac{d}{dt}+S_{11}\right)Q_1+\,..\,+\left(L_{1n}\frac{d^2}{dt^2}+R_{1n}\frac{d}{dt}+S_{1n}\right)Q_n=E_1, \\
\text{- -} \\
\left(L_{n1}\frac{d^2}{dt^2}+R_{n1}\frac{d}{dt}+S_{n1}\right)Q_1+\,..\,+\left(L_{nn}\frac{d^2}{dt^2}+R_{nn}\frac{d}{dt}+S_{nn}\right)Q_n=E_n.
\end{array} \tag{3}
$$

Here, as a matter of convenience, we have written $1/C_{jk}=S_{jk}$. It is often more convenient, at least at the outset, to deal with equations (3) rather than (1).

The Exponential Solution

In taking up the mathematical solution of equations (1), we shall start with the *exponential solution.* This is of fundamental importance, both theoretically and practically. It serves as the most direct introduction to the Heaviside Operational Calculus, and in addition furnishes the basis of the *steady-state* solution, or the theory of alternating currents.

To derive this solution we set $E_1=F_1e^{\lambda t}$ and put all the other forces $E_2, \, .\, . \, E_n$ equal to zero. This latter restriction is a mere matter of convenience, and, in virtue of the linear character of the equations, involves no loss of generality.

Now, corresponding to $E_1=F_1e^{\lambda t}$, let us assume a solution of the form

$$I_j=J_je^{\lambda t} \qquad (j=1,2\,.\,.\,n)$$

where J_j is a constant. So far this is a pure assumption, and its correctness must be verified by substitution in the differential equations.

Now if $I_j=J_je^{\lambda t}$, it follows at once that

$$\frac{d}{dt}I_j=\lambda I_j=\lambda J_je^{\lambda t}$$

and

$$\int I_j dt=\frac{1}{\lambda}I_j=\frac{1}{\lambda}J_je^{\lambda t}.$$

Now substitute these relations in equations (1) and cancel the common factor $e^{\lambda t}$. We then get the system of simultaneous equations

$$
\begin{array}{c}
(\lambda L_{11}+R_{11}+1/\lambda C_{11})J_1+\,.\,.+(\lambda L_{1n}+R_{1n}+1/\lambda C_{1n})J_n=F_1, \\
(\lambda L_{21}+R_{21}+1/\lambda C_{21})J_1+\,.\,.+(\lambda L_{2n}+R_{2n}+1/\lambda C_{2n})J_n=o, \\
\text{- -} \\
(\lambda L_{n1}+R_{n1}+1/\lambda C_{n1})J_1+\,.\,.+(\lambda L_{nn}+R_{nn}+1/\lambda C_{nn})J_n=o.
\end{array} \tag{4}
$$

We note that this is a system of simultaneous *algebraic* equations from which the time factor has disappeared. It is this that makes

the exponential solution so simple, since we can immediately pass from differential equations to algebraic equations. In these algebraic equations, n in number, there are n unknown quantities $J_1, \ldots J_n$. These can therefore all be uniquely determined. We thus see that the assumed form of solution is possible.

The notation of equations (4) may be profitably simplified as follows: write

$$\lambda L_{jk}+R_{jk}+1/\lambda C_{jk}=z_{jk}(\lambda)=z_{jk}$$

and we have

$$\begin{aligned} z_{11}J_1+z_{12}J_2+\ldots+z_{1n}J_n&=F_1, \\ z_{21}J_1+z_{22}J_2+\ldots+z_{2n}J_n&=o, \\ &\cdots \\ z_{n1}J_1+z_{n2}J_2+\ldots+z_{nn}J_n&=o. \end{aligned} \tag{5}$$

The solution of this system of equations is

$$J_j=\frac{M_{j1}(\lambda)}{D(\lambda)}F_1=\frac{M_{j1}}{D}F_1$$

and

$$I_j=\frac{M_{j1}}{D}F_1e^{\lambda t}=\frac{F_1}{Z_{j1}}e^{\lambda t} \tag{6}$$

where D is the determinant of the coefficients,

$$\begin{vmatrix} z_{11} & z_{12} & z_{13} & \ldots & z_{1n} \\ z_{21} & z_{22} & z_{23} & \ldots & z_{2n} \\ z_{31} & z_{32} & \ldots & \ldots & z_{3n} \\ \cdots & & & & \\ z_{n1} & z_{n2} & \ldots & \ldots & z_{nn} \end{vmatrix} \tag{7}$$

and M_{j1} is the cofactor, or minor with proper sign, of the jth column and first row.

I shall not attempt to discuss the theory of determinants on which this solution is based.[1] We may note, however, one important property. Since $z_{jk}=z_{kj}$, $M_{jk}=M_{kj}$. From this the Reciprocal Theorem follows immediately. This may be stated as follows:

If a force $Fe^{\lambda t}$ is applied in the jth mesh, or branch, of the network, the current in the kth mesh, or branch, is by the foregoing

$$\frac{M_{kj}}{D}Fe^{\lambda t}.$$

Now apply the same force in the kth mesh, or branch, then the current in the jth mesh is

$$\frac{M_{jk}}{D}Fe^{\lambda t}.$$

[1] For a remarkably concise and complete discussion of the exponential solution by aid of the theory of determinants, see Cisoidal Oscillations, Trans. A. I. E. E., 1911, by G. A. Campbell.

Comparing these expressions and remembering that $M_{kj}=M_{jk}$, it follows that the current in the kth branch corresponding to an exponential impressed e.m.f. in the jth branch, is equal to the current in the jth branch corresponding to the same e.m.f. in the kth branch. This relation is of the greatest technical importance.

In many important technical problems we are interested only in two accessible branches, such as the sending and receiving. In such cases, where we are not concerned with the currents in the other meshes or branches, it is often convenient to eliminate them from the equation. Thus suppose that we have electromotive forces E_1 and E_2 in meshes 1 and 2 and are concerned only with the currents in these meshes. If we solve equations 3, 4, . . n, $n-2$ in number, for I_3 . . I_n in terms of I_1 and I_2 and then substitute in (1) and (2) we get

$$\begin{aligned} Z_{11}I_1+Z_{12}I_2&=E_1,\\ Z_{21}I_1+Z_{22}I_2&=E_2. \end{aligned} \tag{8}$$

The Steady State Solutions

The steady state solution, on which the whole theory of alternating currents depends, is immediately derivable from the exponential solution. Let us suppose that $E_2=E_3=\ldots=E_n=o$ and that $E_1=F\cos(\omega t-\theta)$. Now by virtue of the well known formula in the theory of the complex variable, $\cos x=\frac{1}{2}e^{ix}+\frac{1}{2}e^{-ix}$, we can write

$$\begin{aligned} E_1&=\tfrac{1}{2}Fe^{i(\omega t-\theta)}+\tfrac{1}{2}Fe^{-i(\omega t-\theta)},\\ &=\tfrac{1}{2}(\cos\theta-i\sin\theta)Fe^{i\omega t}+\tfrac{1}{2}(\cos\theta+i\sin\theta)Fe^{-i\omega t},\\ &=\tfrac{1}{2}F'e^{i\omega t}+\tfrac{1}{2}F''e^{-i\omega t}. \end{aligned} \tag{9}$$

Now, by virtue of this formula, the applied electromotive force E_1 consists of two exponential forces, one varying as $e^{i\omega t}$ and the other as $e^{-i\omega t}$. Hence it is easy to see that the currents are made up of two components, thus

$$I_j=J_j'e^{i\omega t}+J_j''e^{-i\omega t} \quad (j=1,2\ldots n) \tag{10}$$

and we have merely to use the exponential solution given above, substituting for λ, $i\omega$ and $-i\omega$ respectively. That is,

$$J_j'=\frac{1}{2}\frac{F'}{Z_{j1}(i\omega)} \text{ and } J_j''=\frac{1}{2}\frac{F''}{Z_{j1}(-i\omega)}$$

or

$$I_j=\frac{1}{2}\frac{Fe^{-i\theta}}{Z_{j1}(i\omega)}e^{i\omega t}+\frac{1}{2}\frac{Fe^{i\theta}}{Z_{j1}(-i\omega)}e^{-i\omega t}.$$

The second term is the conjugate imaginary of the first, so that

$$\begin{aligned} I_j &= R\frac{Fe^{-i\theta}}{Z_{j1}(i\omega)}e^{-i\omega t} \\ &= R\frac{F}{Z_{j1}(i\omega)}e^{i(\omega t-\theta)} \\ &= R\frac{F}{|Z_{j1}(i\omega)|}e^{i(\omega t-\theta-\phi)} \\ &= \frac{F}{|Z(i\omega)|}\cos(\omega t-\theta-\phi). \end{aligned}$$

We thus arrive at the rule for the steady state solution:

If the applied e.m.f. is $F\cos(\omega t-\theta)$, substitute $i\omega$ for d/dt in the differential equations, determine the impedance function

$$Z(i\omega)=D(i\omega)/M(i\omega) \tag{11}$$

by the solution of the algebraic equations, and write it in the form

$$Z(i\omega)=|Z(i\omega)|e^{i\phi}. \tag{12}$$

Then the required solution is

$$I=\frac{F}{|Z(i\omega)|}\cos(\omega t-\theta-\phi). \tag{13}$$

This in compact form contains the whole theory of the symbolic solution of alternating current problems.

The Complementary Solution

So far in the solutions which we have discussed the currents are of the same type as the impressed forces: that is to say in physical language, the currents are "forced" currents and vary with time in precisely the same manner as do the electromotive forces. Such currents are, however, in general only part of the total currents. In addition to the forced currents we have also the characteristic oscillations; or, in mathematical language, the complete solution must include both particular and complementary solutions. This may be shown as follows: Let I_1', ... I_n' be solutions of the complementary equations,

$$\begin{aligned} &\left(L_{11}\frac{d}{dt}+R_{11}+\frac{1}{C_{11}}\int dt\right)I_1'+\,.\,.+\left(L_{1n}\frac{d}{dt}+R_{1n}+\frac{1}{C_{1n}}\int dt\right)I_n'=o, \\ &\text{-------------------------------------} \\ &\left(L_{n1}\frac{d}{dt}+R_{n1}+\frac{1}{C_{n1}}\int dt\right)I_1'+\,.\,.+\left(L_{nn}\frac{d}{dt}+R_{nn}+\frac{1}{C_{nn}}\int dt\right)I_n'=o. \end{aligned} \tag{14}$$

Then if $I_1 \ldots I_n$ is a solution of (1), $I_1+I_1', \ldots I_n+I_n'$, is also a solution.

To derive the solution of the complementary system of equations (14), assume that a solution exists of the form

$$I_j' = J_j' e^{\lambda t} \qquad (j=1,2 \ldots n)$$

so that $d/dt=\lambda$ and $\int dt=1/\lambda$. Substitute in equations (14) and cancel out the common factor $e^{\lambda t}$. Then we have

$$\begin{array}{c} Z_{11}(\lambda)J_j' + \ldots + Z_{1n}(\lambda)J_n' = o, \\ \text{-----------------------} \\ Z_{n1}(\lambda)J_1' + \ldots + Z_{nn}(\lambda)J_n' = o. \end{array} \tag{15}$$

This is a system of n homogeneous equations in the unknown quantities $J_1', \ldots J_n'$. The condition that a finite solution shall exist is that, in accordance with a well known principle of the theory of equations, the determinant of the coefficients shall vanish. That is,

$$D(\lambda) = \begin{vmatrix} Z_{11}(\lambda) & \ldots & Z_{1n}(\lambda) \\ \text{-----} & \text{-----} & \text{-----} \\ \text{-----} & \text{-----} & \text{-----} \\ Z_{n1}(\lambda) & \ldots & Z_{nn}(\lambda) \end{vmatrix} = o. \tag{16}$$

Consequently the possible values of λ must be such that this equation is satisfied. In other words, λ must be a root of the equation $D(\lambda)=o$. Let these roots be denoted by $\lambda_1, \lambda_2 \ldots \lambda_m$. Then, assigning to λ any one of these values, we can determine the ratio J_j'/J_k' from any $(n-1)$ of the equations. That is to say, if we take

$$I_1' = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \ldots + C_m e^{\lambda_m t}, \tag{17}$$

substitution in any $(n-1)$ of the equations determines $I_2', \ldots I_n'$. The m constants $C_1, \ldots C_m$ are so far, however, entirely arbitrary, and are at our disposal to satisfy imposed *boundary conditions*.

This introduces us to the idea of boundary conditions which is of the greatest importance in circuit theory. In physical language the boundary conditions denote the state of the system when the electromotive force is applied or when any change in the circuit constants occurs. The number of independent boundary conditions which can, in general, be satisfied is equal to the number of roots of the equation $D(\lambda)=o$. Evidently, therefore, it is physically impossible to impose more boundary conditions than this. On the other hand, if this number of boundary conditions is not specified, the complete solution is indeterminate: That is to say, the problem is not correctly set. As an example of boundary conditions, we may specify that the

electromotive force is applied at time $t=o$, and that at this time all the currents in the inductances and all the charges on the condensers are zero.

So far we have been following the classical theory of linear differential equations. We have seen that the forced exponential solution and the derived steady state solution are extremely simple and are mere matters of elementary algebra. The practical difficulties in the classical method of solutions begin with the determination of the constants $C_1, \ldots C_m$ of the complementary solution as well as the roots $\lambda_1, \ldots \lambda_m$ of the equation $D(\lambda)=o$. It is at this point that Heaviside broke with classical methods, and by considering special boundary conditions of great physical importance, and particular types of impressed forces, laid the foundations of original and powerful methods of solution. We shall therefore at this point follow Heaviside's example and attack the problem from a different standpoint. In doing this we shall not at once take up an exposition of Heaviside's own method of attack. We shall first establish some fundamental theorems which are extremely powerful and will serve us as a guide in interpreting and rationalizing the Heaviside Operational Calculus.

CHAPTER II

The Solution when an Arbitrary Force is Applied to the Network in a State of Equilibrium

In engineering applications of electric circuit theory there are three outstanding problems:

(1) The steady state distribution of currents and potentials when the network is energized by a sinusoidal electromotive force. This problem is the subject of the theory of alternating currents which forms the basis of our calculations of power lines and the more elaborate networks of communication systems.

(2) The distribution of currents and potentials in the network in response to an arbitrary electromotive force applied to the network in a state of equilibrium, i.e., applied when the currents and charges in the network are identically zero.

(3) The effect on the distribution of currents and potentials of suddenly changing a circuit constant or connection, such as opening or closing a switch, while the system is energized.

We shall base our further analysis of circuit theory on the solutions of problem (2), for the following reasons:

(A) It is essentially a generalization of the Heaviside problem and its solution will furnish us a key to the correct understanding and

interpretation of operational methods and lead to an auxiliary formula from which the rules of the Operational Calculus are directly deducible.

(B) The solution of problem (2) carries with it the solution of problem (3) and also serves as a basis for the theory of alternating currents.

(C) The solution of problem (2) leads directly to an extension of circuit theory to the case where the network contains variable elements: i.e., circuit elements which vary with time and in which non-linear relations obtain.

Problem (2) is therefore the fundamental problem of circuit theory and the formula which we shall now derive may be termed the fundamental formula of circuit theory.

Consider a network in any branch of which, say branch 1, a unit e.m.f. is inserted at time $t=o$, the network having been previously in equilibrium. By unit e.m.f. is meant an electromotive force which has the value unity for all positive values of time ($t \geqq o$). Let the resultant current in any branch, say branch n, be denoted by $A_{n1}(t)$. $A_{n1}(t)$ will be termed the *indicial admittance* of branch n with respect to branch 1—or, more fully, the transfer indicial admittance.

The indicial admittance, aside from its direct physical significance, plays a fundamental role in the mathematical theory of electric circuits. In words, it may be defined as follows: The indicial admittance, $A_{n1}(t)$, is equal to the ratio of the current in branch n, expressed as a time function, to the magnitude of the steady e.m.f. suddenly inserted at time $t=o$ in branch 1. It is evidently a function which is zero for negative values of time and approaches either zero or a steady value (the d.c. admittance) for all actual dissipative systems, as t approaches infinity. It may be noted that, aside from its mathematical determination, which will engage our attention later, it is an experimentally determinable function.

We note, in passing, an important property of the indicial admittance $A_{jk}(t)$, which is deducible from the reciprocal theorem:[2] this is that $A_{jk}(t)=A_{kj}(t)$. That is to say, the value of the transfer indicial admittance is unchanged by an interchange of the driving point and receiving point. It is therefore immaterial in the expression $A_{jk}(t)$ whether the e.m.f. is inserted in branch j and the current measured in branch k, or vice-versa. In general, unless we are concerned with particular branches, the subscripts will be omitted and we shall simply write $A(t)$, it being understood that any two branches

[2] Exceptions to this relation exist where the network contains sources of energy such as amplifiers. These need not engage our attention here.

or a single branch (for the case of equal subscripts) may be under consideration.

From the linear character of the network, it is evident that if a steady e.m.f. $E=E_\tau$ is inserted at time $t=\tau$, the network being in equilibrium, the resultant current is

$$E_\tau \cdot A(t-\tau).$$

Generalizing still further, suppose that steady e.m.fs. $E_o, E_1, E_2, \ldots E_n$ are impressed in the same branch at the respective times $\tau_o, \tau_1, \tau_2 \ldots \tau_n$; the resultant current is evidently

$$E_oA(t)+E_1A(t-\tau_1)+\ldots+E_nA(t-\tau_n)=\sum_{j=o}^{n}E_jA(t-\tau_j). \tag{18}$$

To apply the foregoing to our problem we suppose that there is applied to the network, initially in a state of equilibrium, an e.m.f. $E(t)$ which has the following properties.

1. It is identically zero for $t<o$.
2. It has the value $E(o)$ for $o\leq t\leq\Delta t$.
3. It has the value $E(o)+\Delta_1E$ for $\Delta t\leq t<2\Delta t$.
4. It has the value $E(o)+\Delta_1E+\Delta_2E$ for $2\Delta t\leq t<3\Delta t$.

In other words it has the increment Δ_jE at time $t=j\Delta t$.

Evidently then the resultant current $I(t)$ is

$$E_oA(t)+\Delta_1EA(t-\Delta t)+\ldots+\Delta_nE.A(t-n\Delta t).$$

Now evidently if the interval Δt is made shorter and shorter, then in the limit $\Delta t\to dt$ and $j\Delta t=\tau$ and

$$\Delta_jE=\frac{d}{d\tau}E(\tau)d\tau.$$

Passing to the limit in the usual manner this summation becomes a definite integral and we get

$$I(t)=E(o)A(t)+\int_0^t A(t-\tau)\frac{d}{d\tau}E(\tau)d\tau. \tag{19}$$

Finally by obvious transformations of the expression we arrive at the fundamental formula of circuit theory

$$I(t)=\frac{d}{dt}\int_0^t A(t-\tau)E(\tau)d\tau, \tag{20}$$

$$=\frac{d}{dt}\int_0^t E(t-\tau)A(\tau)d\tau. \tag{20-a}$$

For completeness we write down the following equivalents of (20) and (20-a)

$$I(t)=A(o)E(t)+\int_0^t A'(t-\tau)E(\tau)d\tau, \tag{20-b}$$

$$=A(o)E(t)+\int_0^t A'(\tau)E(t-\tau)d\tau, \tag{20-c}$$

$$=E(o)A(t)+\int_0^t E'(t-\tau)A(\tau)d\tau, \tag{20-d}$$

$$=E(o)A(t)+\int_0^t E'(\tau)A(t-\tau)d\tau. \tag{20-e}$$

where the primes denote differentiation with respect to the argument. Thus $A'(t)=d/dt\,A(t)$.

These equations are the fundamental formulas which mathematically relate the current to the type of applied electromotive force and the constants and connections of the system, and constitute the first part of the solution of our problem. The most important immediate deductions from these formulas are expressed in the following theorems.

1. The indicial admittance of an electrical network completely determines, within a single integration, the behavior of the network to all types of applied electromotive forces. As a corollary, a knowledge of the indicial admittance is the sole information necessary to completely predict the performance and characteristics of the system, including the steady state.

2. The applied e.m.f. and the inidical admittance are similarly and coequally related to the resultant current in the network. As a corollary the form of the current may be modified either by changing the constants and connections of the network or by modifying the form of the applied e.m.f.

3. Since the applied e.m.f. may be discontinuous these formulas determine not only the building up of the current in response to an applied e.m.f. but also its subsidence to equilibrium when the e.m.f. is removed and the network left to itself. In brief, formulas (20) reduce the whole problem to a determination of the indicial admittance of the network. In addition, as we shall see, they lead directly to an integral equation which determines this function.

It is of interest to show the relation between formulas (20) and the usual steady state equations. To do this let the e.m.f., applied at

time $t=o$, be $E \sin(\omega t+\theta)$. Substitution in formula (20-b) and rearrangement gives

$$
\begin{aligned}
I(t) = {} & A(o)E \sin(\omega t+\theta) \\
& + E \sin(\omega t+\theta)\int_0^t \cos \omega\tau A'(\tau)d\tau \\
& - E \cos(\omega t+\theta)\int_0^t \sin \omega\tau A'(\tau)d\tau
\end{aligned}
\tag{21}
$$

where $\qquad A'(t)=\frac{d}{dt}A(t).$

Now this can be resolved into two parts

$$
\begin{aligned}
& E \sin(\omega t+\theta)\left\{A(o)+\int_0^\infty \cos \omega\tau A'(\tau)d\tau\right\} \\
& - E \cos(\omega t+\theta)\left\{\int_0^\infty \sin \omega\tau A'(\tau)d\tau\right\}
\end{aligned}
\tag{22}
$$

which is the *final steady state*, and

$$
\begin{aligned}
& - E \sin(\omega t+\theta)\int_t^\infty \cos \omega\tau A'(\tau)d\tau \\
& + E \cos(\omega t+\theta)\int_t^\infty \sin \omega\tau A'(\tau)d\tau
\end{aligned}
\tag{23}
$$

which is the *transient distortion*, which ultimately dies away for sufficiently large values of time.

To correlate the foregoing expressions for the steady state with the usual formulas we observe that if the symbolic impedance of the network at frequency $\omega/2\pi$ be denoted by $Z(i\omega)$, and if we write

$$\frac{1}{Z(i\omega)}=\alpha(\omega)+i\beta(\omega)$$

then the steady state current is

$$E[\alpha(\omega).\sin(\omega t+\theta)+\beta(\omega).\cos(\omega t+\theta)].$$

Comparison with (22) gives at once

$$\alpha(\omega)=A(o)+\int_0^\infty \cos \omega\tau\, A'(\tau)d\tau, \tag{24}$$

$$\beta(\omega)=-\int_0^\infty \sin \omega\tau\, A'(\tau)d\tau. \tag{25}$$

The Integral Equation for the Indicial Admittance

So far we have tacitly assumed that the indicial admittance is known. As a matter of fact its determination constitutes the essential part of our problem. It is, in fact, the Heaviside problem, and its investigation, to which we now proceed, will lead us directly to the Operational Calculus.

Heaviside's method in investigating this problem was intuitive and "experimental". We, however, shall establish a very general integral equation from which we shall directly deduce his methods and extensions thereof.

Let us suppose that an e.m.f. e^{pt}, where p is either positive real quantity or complex with real part positive, is suddenly impressed on the network at time $t=o$. It follows from the foregoing theory that the resultant current $I(t)$ will be made up of two parts, (1) a forced exponential part which varies with time as e^{pt}, and (2) a complementary part which we shall denote by $y(t)$. The exponential or "forced" component is simply $e^{pt}/Z(p)$, where $Z(p)$ is functionally of the same form as the usual symbolic or complex impedance $Z(i\omega)$. It is gotten from the differential equations of the problem, as explained in a preceding section, by replacing d^n/dt^n by p^n, cancelling out the common factor e^{pt}, and solving the resulting algebraic equation. The complementary or characteristic component, denoted by $y(t)$, depends on the constants and connections of the network, and on the value of p. It does not, however, contain the factor e^{pt} and it dies away for sufficiently large value of t, in all actual dissipative systems. Thus

$$I(t)=\frac{e^{pt}}{Z(p)}+y(t). \tag{26}$$

Now return to formula (20-a) and replace $E(t)$ by e^{pt}. We get

$$I(t)=\frac{d}{dt}e^{pt}\int_0^t A(\tau)e^{-p\tau}d\tau$$

which can be written as

$$\frac{d}{dt}\left\{e^{pt}\int_0^\infty A(\tau)e^{-p\tau}d\tau-e^{pt}\int_t^\infty A(\tau)e^{-p\tau}d\tau\right\}.$$

Carrying out the indicated differentiation this becomes

$$I(t)=pe^{pt}\int_0^\infty A(\tau)e^{-p\tau}d\tau-pe^{pt}\int_t^\infty A(\tau)e^{-p\tau}d\tau+A(t). \tag{27}$$

Equating the two expressions (26) and (27) for $I(t)$ and dividing through by e^{pt} we get

$$\frac{1}{Z(p)}+y(t)e^{-pt}=p\int_0^\infty A(\tau)e^{-p\tau}d\tau-p\int_t^\infty A(\tau)e^{-p\tau}d\tau+A(t)e^{-pt}. \quad (28)$$

This equation is valid for all values of t. Consequently if we set $t=\infty$, and if the real part of p is positive, only the first term on the right and the left hand side of the equation remain, the rest vanishes, and we get

$$\frac{1}{pZ(p)}=\int_0^\infty A(t)e^{-pt}dt. \quad (29)$$

This is an integral equation[3] *valid for all positive real values of p, which completely determines the indicial admittance $A(t)$.* It is on this equation that we shall base our discussion of operational methods and from which we shall derive the rules of the Operational Calculus. Equations (20) and (29) constitute a complete mathematical formulation of our problem, and from them the complete solution is obtainable without further recourse to the differential equations, or further consideration of boundary conditions.

To summarize the preceding: we have reduced the determination of the current in a network in response to an electromotive force $E(t)$, impressed on the network at reference time $t=o$, to the mathematical solution of two equations: first the integral equation

$$\frac{1}{pZ(p)}=\int_0^\infty A(t)e^{-pt}dt \quad (29)$$

and second, the definite integral

$$I(t)=\frac{d}{dt}\int_0^t A(t-\tau)E(\tau)d\tau. \quad (20)$$

It will be observed that in deducing these equations we have merely postulated (1) the linear and invariable character of the network and (2) the existence of an exponential solution of the type $e^{pt}/Z(p)$ for positive values of p. Consequently, while we have so far discussed these formulas in terms of the determination of the current in a finite network, they are not limited in their application to this specific problem. In this connection it may be well to call attention explicitly to the following points.

[3] An integral equation is one in which the unknown function appears under the sign of integration. (29) is an integral equation of the Laplace type. If $Z(p)$ is specified, $A(t)$ is uniquely determined. Methods for solving the integral equations are considered in detail later, in connection with the exposition of the Operational Calculus. The phrase "all positive values of p" will be understood as meaning all values of p in the right hand half of the complex plane.

The formulas and methods deduced above apply not only to finite networks, involving a finite system of linear equations, but to infinite networks and to transmission lines, involving infinite systems of equations, and partial differential equations: in fact to all electrical and dynamical systems in which the connections and constants are linear and invariable.

Secondly the variable determined by formula (20) and (29) need not, of course, be the current. It may equally well be the charge, potential drop, or any of the variables with which we may happen to be concerned. This fact may be explicitly recognized by writing the formulas as:

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}dt, \tag{30}$$

$$x(t)=\frac{d}{dt}\int_0^t h(t-\tau)E(\tau)d\tau. \tag{31}$$

Here $E(t)$ is the applied e.m.f., $x(t)$ is the variable which we desire to determine (charge, current, potential drop, etc.), and

$$x=E/H(p) \tag{32}$$

is the operational equation. $H(p)$ therefore corresponds to and is determined in precisely the same way as the impedance $Z(p)$, but it may not have the physical significance or the dimensions of an impedance. Similarly in character and function, $h(t)$ corresponds to the indicial admittance, though it may not have the same physical significance. It is a generalization of the indicial admittance and may be appropriately termed the *Heaviside Function.* Similarly $H(p)$ may be termed the *generalized impedance function.*

CHAPTER III

The Heaviside Problem and the Operational Equation

The physical problem which Heaviside attacked and which led to his Operational Calculus was the determination of the response of a network or electrical system to a "unit e.m.f." (zero before, unity after time $t=o$) with, of course, the understanding that the system is in equilibrium when the electromotive force is applied. His problem is therefore, essentially that of the determination of the indicial admittance. In our exposition and critique of Heaviside's method of dealing with this problem we shall accompany an account of his own method of solution with a parallel solution from the corresponding integral equation of the problem.

Heaviside's first step in attacking this problem was to start with the differential equations, and replace the differential operator d/dt by the symbol p, and the operation $\int dt$ by $1/p$, thus reducing the equations to an algebraic form. He then wrote the impressed e.m.f. as 1 (unity), thus limiting the validity of the equations to values of $t \geqq o$. The formal solution of the algebraic equations is straightforward and will be written as

$$h = 1/H(p) \tag{33}$$

where h is the "generalized indicial admittance," or Heaviside function (denoting current, charge, potential or any variable with which we are concerned) and $H(p)$ is the corresponding generalized impedance. Thus, if we are concerned with the current in any part of the network, we write

$$A = 1/Z(p). \tag{34}$$

The more general notation is desirable, however, as indicating the wider applicability of the equation.

The equations

$$h = 1/H(p)$$
$$A = 1/Z(p)$$

are the *Heaviside Operational Equations*. They are, as yet, purely symbolic and we have still the problem of determining their explicit meaning and in particular the significance of the operator p.

Comparison of the Heaviside Operational Equations with the integral equations (29) and (30) of the preceding chapter leads to the following fundamental theorem.

The Heaviside Operational Equations

$$A = 1/Z(p)$$
$$h = 1/H(p)$$

are merely the symbolic or short-hand equivalents of the corresponding integral equations

$$\frac{1}{pZ(p)} = \int_0^\infty A(t) e^{-pt} dt$$

$$\frac{1}{pH(p)} = \int_0^\infty h(t) e^{-pt} dt.$$

The integral equations, therefore, supply us with the meaning and significance of the operational equations, and from them the rules of the Operational Calculus are deducible.

By virtue of this theorem, we have the advantage, at the outset, of a key to the meaning of Heaviside's operational equations, and a means of checking and deducing his rules of solution. This will serve us as a guide throughout our further study.

Returning now to Heaviside's own point of view and method of attack, his reasoning may be described somewhat as follows:—The operational equation

$$h=1/H(p)$$

is the full equivalent of the differential equations of the problem and must therefore contain the information necessary to the solution provided we can determine the significance of the symbolic operator p. The only way of doing this, when starting with the operational equation, is one of induction: that is, we must compare the operational equation with known solutions of specific problems and thus attempt to infer by induction general rules for interpreting the operational equation and converting it into the required explicit solution.

The Power Series Solution

Let us start with the simplest possible problem: the current in response to a "unit e.m.f." in a circuit consisting of an inductance L in series with a resistance R.

The differential equation of the problem is

$$L\frac{d}{dt}A+RA=1, \qquad t\geqq o,$$

where A is the indicial admittance. Consequently replacing d/dt by p, the operational equation is

$$A=\frac{1}{pL+R}.$$

The explicit solution is easily derived: it is

$$A=\frac{1}{R}(1-e^{-\alpha t})$$

where $\alpha=R/L$. Note that this makes the current initially zero, so that the equilibrium boundary condition at $t=o$ is satisfied.

Now suppose that we expand the operational equation in inverse powers of p: we get, formally,

$$A=\frac{1}{pL}\,\frac{1}{1+\alpha/p}=\frac{1}{R}\frac{\alpha}{p}\frac{1}{1+\alpha/p}=\frac{1}{R}\left\{\frac{\alpha}{p}-\left(\frac{\alpha}{p}\right)^2+\left(\frac{\alpha}{p}\right)^3-\left(\frac{\alpha}{p}\right)^4+\,\ldots\right\}$$

by the Binomial Theorem.

Now expand the explicit solution as a power series in t: it is

$$A=\frac{1}{R}\left\{\frac{\alpha t}{1!}-\frac{(\alpha t)^2}{2!}+\frac{(\alpha t)^3}{3!}-\ \ldots\ldots\right\}.$$

Comparing the two expansions we see at once that the operational expansion is converted into the explicit solution by assigning to the symbol $1/p^n$ the value $t^n/n!$. It was from this kind of inductive inference that Heaviside arrived at his power series solution.

Now there are several important features in the foregoing which require comment. In the first place the operational equation is converted into the explicit solution only by a particular kind of expansion, namely an expansion in inverse powers of the operator p. For example, if in the operational equation

$$A=\frac{1}{R}\,\frac{\alpha/p}{1+\alpha/p}$$

we replace $1/p$ by $t/1!$ we get

$$A=\frac{1}{R}\,\frac{\alpha t}{1+\alpha t}$$

which is incorrect. Furthermore, if we expand in ascending instead of descending powers of p, namely

$$A=\frac{1}{R}\left\{1-(p/\alpha)+(p/\alpha)^2-\ \ldots\ldots\right\}$$

no correlation with the explicit solution is possible and no significance can be attached to the expansion. We thus infer the general principle, and we shall find this inference to be correct, that the operational equation is convertible into the explicit solution only by the proper choice of expansion of the impedance function, or rather its reciprocal.

In the second place we notice that in writing down the operational equation and then converting it into the explicit solution no consideration has been given to the question of boundary conditions. This is one of the great advantages of the operational method: the boundary conditions, *provided they are those of equilibrium*, are automatically taken care of. This will be illustrated in the next example:

Let a "unit e.m.f." be impressed on a circuit consisting of resistance R, inductance L, and capacity C: required the resultant charge on the condenser.

The differential equation for the charge Q is

$$\left(L\frac{d^2}{dt^2}+R\frac{d}{dt}+1/C\right)Q=1,\qquad t\geqq o.$$

Consequently the operational formula is

$$Q=\frac{1}{Lp^2+Rp+1/C}$$

$$=\frac{1}{Lp^2}\frac{1}{1+a/p+b/p^2}\text{ where }a=\frac{R}{L}\text{ and }b=\frac{1}{LC}.$$

This can be expanded by the Binomial Theorem as

$$Q=\frac{1}{Lp^2}\left\{1-\left(\frac{a}{p}+\frac{b}{p^2}\right)+\left(\frac{a}{p}+\frac{b}{p^2}\right)^2-\left(\frac{a}{p}+\frac{b}{p^2}\right)^3+\,.\,.\right\}.$$

Performing the indicated operations and collecting in inverse powers of p, the first few terms of the expansion are:—

$$\frac{1}{Lp^2}\left\{1-\frac{c_1}{p}-\frac{c_2}{p^2}+\frac{c_3}{p^3}+\frac{c_4}{p^4}-\frac{c_5}{p^5}-\frac{c_6}{p^6}+\,.\,.\,.\right\}$$

where

$$c_1=a$$
$$c_2=b-a^2$$
$$c_3=2ab-a^3$$
$$c_4=b^2-3a^2b+a^4$$
$$c_5=3ab^2-4a^3b+a^5$$
$$c_6=b^3-6a^2b^2+5a^4b-a^6$$

- - - - - - - - - - - - - - - -

We infer therefore that in accordance with the rule of replacing $1/p^n$ by $t^n/n!$ the solution is:—

$$Q=\frac{1}{L}\left\{\frac{t^2}{2!}-c_1\frac{t^3}{3!}-c_2\frac{t^4}{4!}+c_3\frac{t^5}{5!}+c_4\frac{t^6}{6!}-\,.\,.\right\}.$$

Owing to the complicated character of the coefficients in the expansion, the series cannot be recognized and summed by inspection. If, however, we put $R=o$) then $a=o$, and the series becomes

$$C\left\{\frac{1}{2!}\left(\frac{t}{\sqrt{LC}}\right)^2-\frac{1}{4!}\left(\frac{t}{\sqrt{LC}}\right)^4+\frac{1}{6!}\left(\frac{t}{\sqrt{LC}}\right)^6-\,.\,.\right\}$$

whence

$$Q=C\{1-\cos\,(t/\sqrt{LC})\}.$$

We have still to verify this solution by comparison with the explicit solution of the differential equation. This is of the form

$$Q=C+k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t}$$

where k_1 and k_2 are constants which must be chosen to satisfy the boundary conditions and λ_1, λ_2 are the roots of the equation

$$L\lambda^2+R\lambda+1/C=o.$$

Now since we have two arbitrary constants we satisfy the equilibrium condition by making Q and dQ/dt zero at $t=o$, whence

$$C+k_1+k_2=o,$$
$$\lambda_1 k_1+\lambda_2 k_2=o,$$

and

$$k_1=\lambda_2 C/(\lambda_1-\lambda_2),$$
$$k_2=\lambda_1 C/(\lambda_2-\lambda_1).$$

We have also

$$\lambda_1=-\frac{a}{2}+\sqrt{\left(\frac{a}{2}\right)^2-b},$$

$$\lambda_2=-\frac{a}{2}-\sqrt{\left(\frac{a}{2}\right)^2-b}.$$

Writing down the power series expansion of

$$Q=C+k_1 e^{\lambda_1 t}+k_2 e^{\lambda_2 t},$$

then

$$Q=(C+k_1+k_2)+(k_1\lambda_1+k_2\lambda_2)\frac{t}{1!}$$
$$+(k_1\lambda_1^2+k_2\lambda_2^2)\frac{t^2}{2!}+\ \ldots\ .$$

Introducing the values of k_1, k_2, λ_1, λ_2 given above and comparing with the power series derived from the operational solution we see that they are identical term by term.

This example illustrates two facts. First the power series expansions may be complicated, laborious to derive and of such form that they cannot be recognized and summed by inspection. In fact in arbitrary networks of a large number of meshes or degrees of freedom the evaluation of the coefficients of the power series expansion is extremely laborious.

On the other hand, in such cases, the solution by the classical method presents difficulties far more formidable—in fact insuperable difficulties from a practical standpoint. First there is the location of the roots of the function $H(\lambda)$, which in arbitrary networks is a practical impossibility without a prohibitive amount of labor. Secondly there is the determination of the integration constants to satisfy the imposed boundary conditions: a process, which, while theoretically

straightforward, is actually in practice extremely laborious and complicated. We note these points in passing; a more complete estimate of the value of the power series solution will be made later.

To summarize the preceding: Heaviside, generalizing from specific examples otherwise solvable, arrived at the following rule:—

Expand the right hand side of the operational equation

$$h=1/H(p)$$

in inverse powers of p: thus

$$h \backsim a_o+a_1/p+a_2/p^2+\ldots+a_n/p^n+\ldots.$$

and then replace $\frac{1}{p^n}$ by $t^n/n!$. *The operational equation is thereby converted into the explicit power series solution*:—

$$h=a_o+a_1t/1!+a_2t^2/2!+\ldots+a_nt^n/n!+\ldots \tag{35}$$

As stated above, this rule was arrived at by pure induction and generalization from the known solution of specific problems. It cannot, therefore, theoretically be regarded as satisfactorily established. The rule can, however, be directly deduced from the integral equation

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}dt.$$

To its derivation from this equation we shall now proceed.

First suppose we *assume* that $h(t)$ admits of the power series expansion

$$h_o+h_1t/1!+h_2t^2/2!+\ldots\ldots$$

Substitute this assumed expansion in the integral, and integrate term by term. The right hand side of the integral equation becomes formally

$$h_o/p+h_1/p^2+h_2/p^3+\ldots\ldots$$

by virtue of the formula

$$\int_0^\infty \frac{t^n}{n!}e^{-pt}=\frac{1}{p^{n+1}} \text{ for } p>o.$$

Now expand the left hand side of the integral equation asymptotically in inverse process of p: it becomes

$$a_o/p+a_1/p^2+a_2/p^3+\ldots\ldots$$

where

$$a_o+a_1/p+a_2/p^2+\ldots.$$

is the asymptotic expansion of $1/H(p)$. Comparing the two expansions and making a term by term identification, we see that $h_n = a_n$ and

$$h(t) = a_o + a_1 t/1! + a_2 t^2/2! + \dots .$$

which agrees with the Heaviside formula.

This procedure, however, while giving the correct result has serious defects from a mathematical point of view. For example, the asymtotic expansion of $1/H(p)$ has usually only a limited region of convergence, and it is only in this region that term by term integration is legitimate. Furthermore we have *assumed* the possibility of expanding $h(t)$ in a power series: an assumption to which there are serious theoretical objections, and which, furthermore, is not always justified. A more satisfactory derivation, and one which establishes the condition for the existence of a power series expansion, proceeds as follows:—

Let $1/H(p)$ be a function which admits of the formal asymptotic expansion

$$\sum_o^\infty a_n/p^n$$

and let it include no component which is asymptotically representable by a series all of whose terms are zero, that is a function $\phi(p)$ such that the limit, as $p \to \infty$, of $p^n \phi(p)$ is zero for every value of n. Such a function is e^{-p}. With this restriction understood, start with the integral equation, and integrate by parts: we get

$$\frac{1}{H(p)} = h(o) + \int_0^\infty e^{-pt} h^{(1)}(t) dt$$

where $h^{(n)}(t)$ denotes $d^n/dt^n h(t)$. Now let p approach infinity: in the limit the integral vanishes and by virtue of the asymptotic expansion

$$1/H(p) \backsim \sum_o^\infty a_n/p^n, \tag{36}$$

$1/H(p)$ approaches the limit a_o. Consequently

$$h(o) = a_o.$$

Now integrate again by parts: we get

$$p(1/H(p) - a_o) = h^{(1)}(o) + \int_0^\infty e^{-pt} h^{(2)}(t) dt.$$

Again let p approach infinity: in the limit the left hand side of the equation becomes a_1 and we have

$$h^{(1)}(o) = a_1.$$

Proceeding by successive partial integrations we thus establish the general relation

$$h^{(n)}(o) = a_n.$$

But by Taylor's theorem, the power series expansion of $h(t)$ is simply

$$h(t) = h(o) + h^{(1)}(o)t/1! + h^{(2)}(o)t^2/2! + \ldots\ldots$$

whence, *assuming the convergence of this expansion*, we get

$$h(t) = a_o + a_1 t/1! + a_2 t^2/2! + \ldots = \sum_o^\infty a_n t^n/n! \qquad (35)$$

which establishes the power series solution. It should be carefully noted, however, that it does not establish the convergence of the power series solution. As a matter of fact, however, I know of no physical problem in which $H(p)$ satisfies the conditions for an asymptotic expansion, where the power series solution is not convergent. On the other hand many physical problems exist, including those relating to transmission lines, where a power series solution is not derivable and does not exist.

The process of expanding the operational equation in such a form as to permit of its being converted into the explicit solution is what Heaviside calls "algebrizing" the equation. In the case of the power series solution the process of algebrizing consists in expanding the reciprocal of the impedance function in an asymptotic series, thus

$$1/H(p) \backsim a_o + a_1/p + a_2/p^2 + \ldots\ldots$$

Regarded as an expansion in the variable p, instead of as a purely symbolic expansion, this series has usually only a limited region of convergence. This fact need not bother us, however, as the series we are really concerned with is

$$a_o + a_1 t/1! + a_2 t^2/2! + \ldots.$$

It is interesting to note in passing that the latter series is what Borel, the French mathematician, calls the *associated function* of the former, and is extensively employed by him in his researches on the summability of divergent series.

The process of "algebrizing," as in the examples discussed above, may often be effected by a straight forward binomial expansion.

In other cases the form of the generalized impedance function $H(p)$ will indicate by inspection the appropriate procedure. A general process, applicable in all cases where a power series exists, is as follows. Write

$$1/H(p)=1/H\left(\frac{1}{x}\right)=G(x). \tag{36}$$

Now expand $G(x)$ as a Taylor's series: thus formally

$$G(x)=G(o)+G^{(1)}(o)\frac{x}{1!}+G^{(2)}(o)\frac{x^2}{2!}+\ldots$$

where

$$G^{(n)}(o)=\left[\frac{d^n}{dx^n}G(x)\right]_{x=o}. \tag{37}$$

Denote $\frac{G^{(n)}(o)}{n!}$ by a_n, replace x^n by $1/p^n$, and we have

$$G(x)=1/H(p)=a_o+a_1/p+a_2/p^2+\ldots$$

This process of "algebrizing" is formally straightforward and always possible. As implied above, however, in many problems much shorter modes of expansion suggest themselves from the form of the function $H(p)$.

We note here, in passing, that the necessary and sufficient conditions for the existence of a power series solution is the possibility of the formal expansion of $G(x)$ as a power series in x.

At this point a brief critical estimate of the scope and value of the power series solution may be in order. As stated above, in a certain important class of problems relating to transmission lines, a power series does not exist, though a closely related series in fractional powers of t may often be derived. Consequently the power series solution is of restricted applicability. Where, however, a power series does exist, in directness and simplicity of derivation it is superior to any other form of solution. Its chief defect, and a very serious defect indeed, is that except where the power series can be recognized and summed, it is usually practically useless for computation and interpretation except for relatively small values of the time t. This disadvantage is inherent and attaches to all power series solutions. For this reason I think Heaviside overestimated the value of power series as practical or working solutions, and that some of his strictures against orthodox mathematicians and their solutions may be justly urged against the power series solution. He was quite right in insisting that a solution must be capable of either interpretation or computation and quite right in ridiculing those formal

solutions which actually conceal rather than reveal the significance of the original differential equations of the problem. On the other hand, the following remark of his indicates to me that Heaviside has a quite exaggerated idea of the value and fundamental character of power series in general: "I regret that the result should be so complicated. But the only alternatives are other equivalent infinite series, or else a definite integral which is of no use until it is evaluated, when the result must be the series (135), or an equivalent one." As a matter of fact the properties of most of the important functions of mathematical physics have been investigated and their values computed by methods other than series expansions. I may add that in technical work the power series solution has proved to be of restricted utility, while definite integrals, which Heaviside [4] particularly despised, have proved quite useful.

The Expansion Theorem Solution [5]

We pass now to the consideration of another extremely important form of solution. Heaviside gives this solution without proof: we shall therefore merely state the solution and then derive it from the integral equation.

Given the operational equation

$$h = 1/H(p)$$

which has the significance discussed above: i.e., the response of the network to a "unit e.m.f.". The explicit solution may be written as

$$h = \frac{1}{H(o)} + \sum_{1}^{n} \frac{e^{p_k t}}{p_k H'(p_k)} \tag{38}$$

where $p_1, p_2 \ldots p_n$ are the n roots of the equation

$$H(p) = o$$

and

$$H'(p_k) = \left[\frac{d}{dp}H(p)\right]_{p=p_k}. \tag{39}$$

As remarked above, this solution, referred to by him as *The Expansion Theorem*, was stated by Heaviside without proof; how he arrived at it will probably always remain a matter of conjecture. Its derivation from the integral equation is, however, a relatively simple matter, though in special cases troublesome questions arise.

[4] Vide a remark of his to the effect that some mathematicians took refuge in a definite integral and called that a solution.

[5] This terminology is due to Heaviside. A more appropriate and physically significant expression would be "The Solution in terms of normal or characteristic vibrations."

The derivation of the expansion solution from the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt$$

follows immediately from the partial fraction expansion

$$\frac{1}{pH(p)} = \frac{1}{pH(o)} + \sum_{j=1}^{n} \frac{1}{(p-p_j)p_jH'(p_j)} \tag{40}$$

where $p_1, p_2 \ldots p_n$ are the roots of the equation $H(p)=o$, and

$$H'(p_j) = \left\{ \frac{d}{dp} H(p) \right\}_{p=p_j}. \tag{41}$$

Partial fraction expansions of this type are fully discussed in treatises on algebra and the calculus and the conditions for their existence established. Before discussing the restrictions imposed on $H(p)$ by this expansion, we shall first, assuming its existence, derive the expansion theorem solution.

By virtue of (40) the integral equation is

$$\frac{1}{pH(o)} + \sum_{1}^{n} \frac{1}{(p-p_j)p_jH'(p_j)} = \int_0^\infty h(t)e^{-pt}dt. \tag{42}$$

The expansion on the left hand side suggests a corresponding expansion on the right hand side: that is, we suppose that

$$h(t) = h_o(t) + h_1(t) + h_2(t) + \ldots + h_n(t) \tag{43}$$

and specify that these component functions shall satisfy the equations

$$\frac{1}{pH(o)} = \int_0^\infty h_o(t)e^{-pt}dt \tag{44}$$

$$\frac{1}{(p-p_j)p_jH'(p_j)} = \int_0^\infty h_j(t)e^{-pt}dt \qquad j=1,2\ldots n. \tag{45}$$

It follows at once from (43) and direct addition of equations (44) and (45) that (42) is satisfied and hence is solved provided $h_o, \ldots h_n$ can be evaluated from (44) and (45).

Now since

$$\int_0^\infty e^{\lambda t}e^{-pt}dt = \frac{1}{p-\lambda} \tag{46}$$

provided the real part of λ is not positive (a condition satisfied in all network problems), we see at once that equations (42) and (43) are satisfied by taking

$$h_o(t)=h_o=\frac{1}{H(o)}, \tag{47}$$

$$h_j(t)=\frac{e^{p_jt}}{p_jH'(p_j)}, \qquad j=1,2, \ldots n.$$

Consequently from (43) and (47) it follows that

$$h(t)=\frac{1}{H(o)}+\sum_1^n \frac{e^{p_jt}}{p_jH'(p_j)} \tag{48}$$

which establishes the Expansion Theorem Solution.

As implied above, the partial fraction expansion (40), on which the expansion theorem solution depends, imposes certain restrictions on the impedance function $H(p)$. Among these are that $H(p)$ must have no zero root, no repeated roots, and $1/H(p)$ must be a proper fraction. In all finite networks these conditions are satisfied, or by a slight modification, the operational equation can be reduced to the required form. The case of repeated roots, which may occur where the network involves a unilateral source of energy such as an amplifier, can be dealt with by assuming unequal roots and then letting the roots approach equality as a limit. Without entering upon these questions in detail, however, we can very simply and directly establish the proposition that the expansion theorem gives the solution whenever a solution in terms of normal or characteristic vibrations exists. The proof of this proposition proceeds as follows.

It is known from the elementary theory of linear differential equations that the general solution of the set of differential equations, of which the operational equation is $h=1/H(p)$, is of the form

$$h(t)=C_o+\sum_1^n C_je^{p_jt}$$

where p_j is the jth root of $H(p)=o$, and $C_o, C_1 \ldots C_n$ are constants of integration which must be so chosen as to satisfy the system of differential equations and the imposed boundary conditions. The summation is extended over all the roots of $H(p)$ which is supposed not to have a zero root or repeated roots.

Now substitute this known form of solution in the integral equation of the problem and carry out the integration term by term. We get

$$\frac{1}{H(p)}=C_o+p\sum\frac{C_j}{p-p_j}. \tag{49}$$

Setting $p=o$, we have at once

$$C_o=1/H(o). \tag{50}$$

To determine C_j let $p=p_j+q$ where q is a small quantity ultimately to be set equal to zero, and write the equation as

$$C_oH(p)+\sum\frac{pH(p)}{p-p_j}C_j=1. \tag{51}$$

If now $p=p_j+q$ and q approaches zero, this becomes in the limit

$$p_jH'(p_j)C_j=1 \tag{52}$$

or

$$C_j=\frac{1}{p_jH'(p_j)}, \tag{53}$$

whence

$$h(t)=\frac{1}{H(o)}+\sum\frac{e^{p_jt}}{p_jH'(p_j)} \tag{54}$$

which is the Expansion Theorem Solution.

We shall not attempt to discuss here cases where the expansion solution breaks down though such cases exist. In every such case, however, the breakdown is due to the failure of the impedance function $H(p)$ to satisfy the conditions necessary for the partial fraction expansion (40), and correlatively the non-existence of a solution in normal vibrations. Furthermore, it is usually possible by simple modification to deduce a modified expansion solution. It may be added here, that while the proof given above is also limited implicitly to finite networks, the expansion solution is valid in most transmission line problems.

Let us now illustrate how the expansion solution works by applying it to a few simple examples. Take first the case considered in the preceding chapter in connection with the power series solution. Required the charge Q on a condenser C in series with an inductance L and resistance R in response to a "unit e.m.f." The operational equation is

$$Q=\frac{1}{Lp^2+Rp+1/C}$$

or

$$Q=\frac{1}{L}\frac{1}{p^2+2\alpha p+\omega^2}$$

where $\alpha = R/2L$ and $\omega^2 = 1/LC$.

The roots of the equation $H(p) = o$ are the roots of the equation

$$p^2 + 2\alpha p + \omega^2 = o$$

whence

$$p_1 = -\alpha + \sqrt{\alpha^2 - \omega^2} = -\alpha + \beta,$$

$$p_2 = -\alpha - \sqrt{\alpha^2 - \omega^2} = -\alpha - \beta.$$

Also $H'(p) = 2L(p + \alpha)$, so that

$$H'(p_1) = 2\beta L$$

$$H'(p_2) = -2\beta L$$

and

$$1/H(o) = 1/L\omega^2 = C.$$

Inserting these expressions in the Expansion Theorem Solution (38), we get

$$Q = C - \frac{e^{-\alpha t}}{2\beta L}\left(\frac{e^{\beta t}}{\alpha - \beta} - \frac{e^{-\beta t}}{\alpha + \beta}\right).$$

It is now easy to verify the fact that this solution satisfies the differential equations and the boundary condition $Q = o$ and $dQ/dt = o$ at time $t = o$.

If $\omega > \alpha$, β is a pure imaginary

$$\beta = i\omega\sqrt{1 - (\alpha/\omega)^2} = i\omega'$$

and

$$Q = C - \frac{e^{-\alpha t}}{\omega' L} \frac{\omega' \cos \omega' t + \alpha \sin \omega' t}{\alpha^2 + \omega'^2}.$$

In connection with this problem we note two advantages of the expansion solution, as compared with the power series solution: (1) it is much simpler to derive from the operational equation, and (2) its numerical computation is enormously easier. A table of exponential and trigometric functions enables us to evaluate Q for any value of t almost at once whereas in the case of the power series solution the labor of computation for large values of t is very great. A third and very important advantage of the expansion solution in this particular problem is that without detailed computation we can deduce by mere inspection the general character of the function and the effect of the circuit parameters on its form: an advantage which never attaches to the power series solution.

This last property of the particular solution above is extremely important. The ideal form of solution, particularly in technical

problems, is one which permits us to infer the general character and properties of the function and the effect of the circuit constants on its form, without detailed solutions. A solution which possesses these properties, even if its exact computation is not possible without prohibitive labor, is far superior to a solution which, while completely computable, tells us nothing without detailed computation. It is for this reason that some of the derived forms of solution, discussed later, are of such importance. In fact a solution which requires detailed computation before it yields the information implied in it is merely equivalent to an experimentally determined solution.

Unfortunately the advantages attaching to the expansion solution of the specific problem just discussed, do not, in general, characterize the expansion solution. The following disadvantages should be noted. First, the location of the roots of the impedance function $H(p)$ is practically impossible in the case of arbitrary networks of more than a few degrees of freedom. In the second place, when the number of degrees of freedom is large it is not only impossible to deduce the significance of the solution by inspection, but the computation becomes extremely laborious. In such cases, the practical value of the expansion solution depends, just as in the power series solution, on the possibility of recognizing and summing the expansion. This will be clear in the case of transmission lines, where the roots of $H(p)$ are infinite in number and the direct computation of the expansion solution (except in the case of the non-inductive cable) is quite impossible.

CHAPTER IV

Some General Formulas and Theorems for the Solution of Operational Equations

We have seen that the operational equation

$$h = 1/H(p)$$

is the symbolic or short-hand equivalent of the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt$$

and from the latter we have deduced two very important forms of the Heaviside solution. In recognizing the equivalence of these two equations we have a very great advantage and are able, in fact, to base the Operational Calculus on deductive instead of inductive

reasoning. In this chapter we shall employ this equivalence to establish certain general formulas and theorems for the solution of operational equations. That is to say, we shall make use of the principles that (1) any method applicable to the solution of the integral equation supplies us with a corresponding method for the solution of the operational equation, and (2) a solution of any specific integral equation gives at once the solution of the corresponding operational equation. We turn therefore to a brief discussion of the appropriate methods for solving the integral equation.

It may be said at the outset, that the solution of the integral equation, like the evaluation of integrals, is a matter of considerable art and experience; in other words there is not, in general, a straightforward procedure corresponding to the process of differentiation.

On the other hand, as a purely mathematical question, it is always possible to invert the integral equation and write down $h(t)$ as an explicit function in the form of an infinite integral. For example it may be shown from the Fourier Integral that

$$h(t)=\frac{2}{\pi}\int_0^\infty \frac{\alpha(\omega)}{\omega}\sin t\omega.d\omega$$

where $\alpha(\omega)$ is defined by

$$\frac{1}{H(i\omega)}=\alpha(\omega)+i\,\beta(\omega).$$

Later on we shall briefly consider the Fourier Integral; for the present the preceding formula will not be considered further. In certain problems it is of value; for the explicit derivation of $h(t)$, however, it is usually too complicated to be of any use except in the hands of professional mathematicians. As a matter of fact, a direct attack on this formula would be equivalent to abandoning the unique simplicity and advantages of the whole Operational Calculus.

It has been noted above that any solution of the integral equation supplies a solution of the corresponding operational equation. This principle enables us to take advantage of the fact that a very large number of infinite integrals of the type

$$\int_0^\infty f(t)e^{-pt}dt$$

have been evaluated. *The evaluation of every infinite integral of this type supplies us, therefore, with the solution of an operational equation.*

Of course, not all the operational equations so solvable have physical significance. Many, however, do. Below is a list of infinite integrals

with their known solutions, accompanied by the corresponding operational equation and its explicit solution. All of these solutions are directly applicable to important technical problems. It may be remarked in passing that the infinite integrals have for the most part been evaluated by advanced mathematical methods which need not concern us here.

Table of Infinite Integrals, the Corresponding Operational Equations, and Their Explicit Solutions

(a) $$\int_0^\infty e^{-pt}e^{-\lambda t}dt=\frac{1}{p+\lambda},$$

$$h=\frac{p}{p+\lambda}=e^{-\lambda t}.$$

(b) $$\int_0^\infty e^{-pt}\frac{t^n}{n!}dt=1/p^{n+1},$$

$$h=\frac{1}{p^n}=t^n/n!.$$

(c) $$\int_0^\infty e^{-pt}\frac{1}{\sqrt{\pi t}}dt=\frac{1}{\sqrt{p}},$$

$$h=\sqrt{p}=1/\sqrt{\pi t}.$$

(d) $$\int_0^\infty e^{-pt}\frac{(2t)^n}{1.3.5\ldots(2n-1)}\frac{dt}{\sqrt{\pi t}}=\frac{1}{p^n\sqrt{p}},$$

$$h=\frac{\sqrt{p}}{p^n}=\frac{(2t)^n}{1.3.5\ldots(2n-1)}\frac{1}{\sqrt{\pi t}}.$$

(e) $$\int_0^\infty e^{-pt}\frac{t^n}{n!}e^{-\lambda t}\,dt=\frac{1}{(p+\lambda)^{n+1}},$$

$$h=\frac{p}{(p+\lambda)^{n+1}}=\frac{t^n}{n!}e^{-\lambda t}.$$

(f) $$\int_0^\infty e^{-pt}\sqrt{\frac{\lambda}{\pi}}\frac{e^{-\lambda/t}}{t\sqrt{t}}dt=e^{-2\sqrt{\lambda p}},$$

$$h=pe^{-2\sqrt{\lambda p}}=\sqrt{\frac{\lambda}{\pi}}\frac{e^{-\lambda/t}}{t\sqrt{t}}.$$

(g) $$\int_0^\infty e^{-pt}\frac{e^{-\lambda/t}}{\sqrt{\pi t}}dt=\frac{e^{-2\sqrt{\lambda p}}}{\sqrt{p}},$$

$$h=\sqrt{p}\,e^{-2\sqrt{\lambda p}}=\frac{e^{-\lambda/t}}{\sqrt{\pi t}}.$$

(h) $$\int_0^\infty e^{-pt} \sin \lambda t\, dt = \frac{\lambda}{p^2+\lambda^2},$$

$$h = \frac{p\lambda}{p^2+\lambda^2} = \sin \lambda t.$$

(i) $$\int_0^\infty e^{-pt} \cos \lambda t\, dt = \frac{p}{p^2+\lambda^2},$$

$$h = \frac{p^2}{p^2+\lambda^2} = \cos \lambda t.$$

(j) $$\int_0^\infty e^{-pt} e^{-\mu t} \cos \lambda t\, dt = \frac{p+\mu}{(p+\mu)^2+\lambda^2},$$

$$h = \frac{p^2+\mu p}{(p+\mu)^2+\lambda^2} = e^{-\mu t} \cos \lambda t.$$

(k) $$\int_0^\infty e^{-pt} e^{-\mu t} \sin \lambda t\, dt = \frac{\lambda}{(p+\mu)^2+\lambda^2},$$

$$h = \frac{p\lambda}{(p+\mu)^2+\lambda^2} = e^{-\mu t} \sin \lambda t.$$

(l) $$\int_0^\infty e^{-pt} J_o(\lambda t) dt = \frac{1}{\sqrt{p^2+\lambda^2}},$$

$$h = \frac{p}{\sqrt{p^2+\lambda^2}} = J_o(\lambda t).$$

(m) $$\int_\lambda^\infty e^{-pt} J_o(\sqrt{t^2-\lambda^2})\, dt = \frac{e^{-\lambda\sqrt{p^2+1}}}{\sqrt{p^2+1}},$$

$$h = \frac{p}{\sqrt{p^2+1}} e^{-\lambda\sqrt{p^2+1}} = o \text{ for } t < \lambda$$

$$= J_o(\sqrt{t^2-\lambda^2}) \text{ for } t \geq \lambda.$$

(n) $$\int_0^\infty e^{-pt} J_n(\lambda t) dt = \frac{1}{r}\left(\frac{r-p}{\lambda}\right)^n, \quad r^2 = p^2+\lambda^2,$$

$$h = \frac{p}{r}\left(\frac{r-p}{\lambda}\right)^n = J_n(\lambda t).$$

(p) $$\int_0^\infty e^{-pt} e^{\lambda t} I_o(\lambda t) dt = \frac{1}{\sqrt{p^2+2\lambda p}},$$

$$h = \frac{1}{\sqrt{1+2\lambda/p}} = e^{-\lambda t} I_o(\lambda t).$$

In formulas (l), (m), (n), $J_n(x)$ denotes the Bessel function of order n and argument x. In formula (p), $I_o(x)$ denotes the Bessel function $J_o(ix)$ where $i=\sqrt{-1}$.

This list might be greatly extended. As it is, we are in possession of a set of solutions of operational equations which occur in important technical problems and which will be employed later.

The foregoing emphasize the practical and theoretical importance of recognizing the equivalence of the integral and operational equations. With this equivalence in mind, the solution of an operational equation is often reduced to a mere reference to a table of infinite integrals. Heaviside did not recognize this equivalence. As a consequence many of his solutions of transmission line problems are extremely laborious and involved and in the end unsatisfactory because expressed in involved power series.

Not all the infinite integrals corresponding to the operational equations of physical problems have been evaluated or can be recognized without transformation. This statement corresponds exactly with the fact that a table of integrals is not always sufficient but must be supplemented by general methods of integration. We turn, therefore, to stating and discussing some general Theorems applicable to the solution of Operational Equations.

In the derivation of the operational theorems, which constitute the general rules of the Operational Calculus, the following proposition, due to Borel and known as Borel's theorem, will be frequently employed.*

If the functions $f(t)$, $f_1(t)$, and $f_2(t)$ are defined by the integral equations

$$F(p)=\int_0^\infty f(t)e^{-pt}dt$$

$$F_1(p)=\int_0^\infty f_1(t)e^{-pt}dt$$

$$F_2(p)=\int_0^\infty f_2(t)e^{-pt}dt$$

and if the functions F, F_1 and F_2 satisfy the relation

$$F(p)=F_1(p).F_2(p)$$

* For a proof of this important theorem the reader is referred to Borel, "Lecons sur les Séries Divergentes" (1901), p. 104; to Bromwich, "Theory of Infinite Series," pp. 280–281; or to Ford, "Studies on Divergent Series and Summability," pp. 93–94 (being Vol. 11 of the Michigan University Science Series, published by Macmillan). The proof depends on Jacobi's transformation of a double integral: see Edward's "Integral Calculus," 1922, Vol. II, pp. 14–15.

then

$$f(t)=\int_0^t f_1(\tau)f_2(t-\tau)d\tau$$
$$=\int_0^t f_2(\tau)f_1(t-\tau)d\tau.$$

The operational theorems will now be stated and briefly proved from the integral equation identity.

Theorem I

If in the Operational Equation

$$h=1/H(p)$$

the generalized impedance function $H(p)$ can be expanded in a sum of terms, thus

$$\frac{1}{H(p)}=\frac{1}{H_1(p)}+\frac{1}{H_2(p)}+\ .\ .\ .\ +\frac{1}{H_n(p)},$$

and if the auxiliary operational equations

$$h_1=\frac{1}{H_1(p)}$$
$$h_2=\frac{1}{H_2(p)}$$
$$-\ -\ -\ -\ -\ -\ -$$

can be solved, then

$$h=h_1+h_2+.\ .\ .\ +h_n.$$

This theorem is too obvious to require detailed proof: in fact it is self evident. The power series and expansion theorem solutions are examples of its application. In general, however, the appropriate form of expansion of $1/H(p)$ will depend on the particular problem in hand. The theorem, as it stands is a formal statement of the fact that solutions can often be obtained by an appropriate expansion whereas the equation cannot be solved as it stands.

Theorem II

If $h=h(t)$ and $g=g(t)$ are defined by the operational equations

$$h=1/H(p)$$
$$g=1/pH(p)$$

then

$$g(t)=\int_0^t h(\tau)d\tau.$$

To prove this theorem we start with the integral equations

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt,$$

$$\frac{1}{p^2H(p)} = \int_0^\infty g(t)e^{-pt}dt.$$

The second of these is in form for an immediate application of Borel's theorem since

$$\frac{1}{p^2H(p)} = \frac{1}{p} \cdot \frac{1}{pH(p)}.$$

The functions f_1 and f_2 of Borel's theorem then satisfy the equations

$$\frac{1}{p} = \int_0^\infty f_1(t)e^{-pt}dt,$$

$$\frac{1}{pH(p)} = \int_0^\infty f_2(t)e^{-pt}dt.$$

It follows at once that

$$f_1(t) = 1$$
$$f_2(t) = h(t)$$

whence by Borel's theorem

$$g(t) = \int_0^t h(\tau)d\tau.$$

Theorem III

If $h = h(t)$ and $g = g(t)$ are defined by the operational equations

$$h = 1/H(p)$$
$$g = p/H(p)$$

then

$$g(t) = \frac{d}{dt}h(t)$$

provided $h(o) = o$.

The integral equations of the problem are

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt,$$

$$\frac{1}{H(p)} = \int_0^\infty g(t)e^{-pt}dt.$$

Integrating the first of these by parts we have,

$$\frac{1}{pH(p)}=\frac{1}{p}h(o)+\frac{1}{p}\int_0^\infty h'(t)e^{-pt}dt$$

where $h'(t)=d/dt\ h(t)$.

If $h(o)=o$, we have at once

$$\frac{1}{H(p)}=\int_0^\infty h'(t)e^{-pt}dt.$$

Comparison with the integral equation for $g(t)$ shows at once that $g(t)=h'(t)$, since the integral equation determines the function uniquely.

Theorems II and III establish the characteristic Heaviside Operations of replacing $1/p$ by $\int_0^t dt$ and p by d/dt.

Theorem IV

If in the operational equation

$$h=1/H(p)$$

the generalized impedance function can be factored in the form

$$H(p)=H_1(p)\cdot H_2(p)$$

and if the auxiliary operational equations

$$h_1=1/H_1(p)$$

$$h_2=1/H_2(p)$$

define the auxiliary variables h_1 and h_2, then

$$h(t)=\frac{d}{dt}\int_0^t h_1(\tau)h_2(t-\tau)d\tau$$

$$=\frac{d}{dt}\int_0^t h_2(\tau)h_1(t-\tau)d\tau.$$

This theorem is immediately deducible from Borel's theorem and theorems II and III, as follows.

The integral equations are

$$\frac{1}{pH(p)}=p\frac{1}{pH_1(p)}\cdot\frac{1}{pH_2(p)}=\int_0^\infty h(t)e^{-pt}dt$$

$$\frac{1}{pH_1(p)}=\int_0^\infty h_1(t)e^{-pt}dt$$

$$\frac{1}{pH_2(p)}=\int_0^\infty h_2(t)e^{-pt}dt.$$

Now define an auxiliary function $g(t)$ by the operational equation

$$g=\frac{1}{pH(p)}.$$

Then

$$\frac{1}{pH_1(p)}\cdot\frac{1}{pH_2(p)}=\int_0^\infty g(t)e^{-pt}dt$$

and by Borel's theorem

$$g(t)=\int_0^t h_1(\tau)h_2(t-\tau)d\tau$$

$$=\int_0^t h_2(\tau)h_1(t-\tau)d\tau.$$

From this equation it follows that $g(o)=o$, and hence comparing the operational equations for h and g, we have by aid of Theorem III

$$h(t)=\frac{d}{dt}g(t)$$

and hence

$$h(t)=\frac{d}{dt}\int_0^t h_1(\tau)h_2(t-\tau)d\tau$$

$$=\frac{d}{dt}\int_0^t h_2(\tau)h_1(t-\tau)d\tau.$$

This theorem is extremely important, although not stated or employed by Heaviside himself. We shall make use of it in establishing two important general theorems and shall have frequent occasion to employ it in specific problems occurring in connection with the subsequent discussion of transmission theory.

Theorem V

If $h=h(t)$ and $g=g(t)$ are defined by the operational equations

$$h=\frac{1}{H(p)}$$

$$g=\frac{1}{H(p+\lambda)}$$

where λ is a positive real parameter, then

$$g(t)=(1+\lambda\int_0^t dt)e^{-\lambda t}h(t).$$

To prove this theorem we start with the integral equations

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}dt$$

$$\frac{1}{pH(p+\lambda)}=\int_0^\infty g(t)e^{-pt}dt.$$

In the first of these equations replace the symbol p by $q+\lambda$: we get

$$\frac{1}{q+\lambda}\cdot\frac{1}{H(q+\lambda)}=\int_0^\infty h(t)e^{-\lambda t}e^{-qt}dt$$

and then to preserve our original notation replace the symbol q by p, whence

$$\frac{1}{(p+\lambda)H(p+\lambda)}=\int_0^\infty h(t)e^{-\lambda t}e^{-pt}dt. \tag{a}$$

The integral equation in $g(t)$ can be written as

$$\left(1+\frac{\lambda}{p}\right)\frac{1}{(p+\lambda)H(p+\lambda)}=\int_0^\infty g(t)e^{-pt}dt. \tag{b}$$

Comparing equations (a) and (b) it follows at once from theorems I and II that

$$g(t)=\left(1+\lambda\int_0^t dt\right)h(t)e^{-\lambda t}.$$

From the foregoing, the following auxiliary theorem is immediately deducible.

Theorem Va

If $h=h(t)$ and $g=g(t)$ are defined by the operational equations

$$h=\frac{1}{H(p)}$$

$$g=\frac{p}{(p+\lambda)H(p+\lambda)}$$

then

$$g(t)=h(t)e^{-\lambda t}.$$

The proof of this theorem will be left as an exercise to the reader.

Theorem VI

If $h=h(t)$ and $g=g(t)$ are defined by the operational equations

$$h=1/H(p)$$

$$g=1/H(\lambda p)$$

where λ is a positive real parameter, then

$$g(t)=h(t/\lambda).$$

We start with the integral equations

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}dt$$

$$\frac{1}{pH(\lambda p)}=\int_0^\infty g(t)e^{-pt}dt$$

and in the first of these equations we replace p by λq and t by τ/λ, whence it becomes

$$\frac{1}{qH(\lambda q)}=\int_0^\infty h\left(\frac{\tau}{\lambda}\right)e^{-q\tau}d\tau.$$

Now replacing the symbols q and τ by p and t respectively, we have

$$\frac{1}{pH(\lambda p)}=\int_0^\infty h(t/\lambda)e^{-pt}dt$$

whence by comparison with the integral equation in $g(t)$ it follows at once that

$$g(t)=h(t/\lambda).$$

This theorem is often useful in making a convenient change in the time scale and eliminating superfluous constants.

Theorem VII

If $h=h(t)$ and $g=g(t)$ are defined by the operational equations

$$h=\frac{1}{H(p)}$$

$$g=\frac{e^{-\lambda p}}{H(p)}$$

where λ is a positive real quantity, then

$$\begin{aligned} g(t) &= o \text{ for } t<\lambda \\ &= h(t-\lambda) \text{ for } t\geq\lambda. \end{aligned}$$

This is a very important theorem in connection with transmission line problems where retardation, due to finite velocity of propagation, occurs. Its proof proceeds as follows:

If the auxiliary function $k=k(t)$ is defined by the operational equation

$$k=e^{-\lambda p}$$

then by Theorem IV,

$$g(t)=\frac{d}{dt}\int_0^t k(\tau)h(t-\tau)d\tau. \tag{a}$$

Now, corresponding to the operational equation $k=e^{-\lambda p}$ we have the integral equation

$$\frac{e^{-\lambda p}}{p}=\int_0^\infty k(t)e^{-pt}dt.$$

The solution of this integral equation, which is easily verified by direct substitution in the infinite integral, is

$$\begin{aligned} k(t)&=o \text{ for } t<\lambda \\ &=1 \text{ for } t\geq\lambda. \end{aligned}$$

Hence equation (a) becomes

$$\begin{aligned} g(t)&=o \text{ for } t<\lambda \\ &=\frac{d}{dt}\int_\lambda^t h(t-\tau)d\tau \text{ for } t\geq\lambda \\ &=h(t-\lambda) \text{ for } t\geq\lambda. \end{aligned}$$

Theorem IV, employed in the preceding proof, as stated above, is extremely important and we shall have frequent occasion to employ it in specific problems. We shall now apply it to deduce an important theorem which extends the operational calculus to arbitrary impressed forces, whereas heretofore the operational equation $h=1/H(p)$ applied only to the case of a "unit e.m.f." impressed on the system.

It will be recalled from a previous chapter that if $x(t)$ denotes the response of a network to an arbitrary force $f(t)$, impressed at time $t=o$, and if $h(t)$ denotes the corresponding response to a "unit e.m.f.," then

$$x(t)=\frac{d}{dt}\int_0^t h(\tau)f(t-\tau)d\tau \tag{31}$$

and

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}dt. \tag{30}$$

Now $f(t)$ may be of such form that the infinite integral

$$\int_0^\infty f(t)e^{-pt}dt$$

can be evaluated and has the value $F(p)/p$: thus

$$\int_0^\infty f(t)e^{-pt}dt = \frac{1}{p}F(p). \tag{55}$$

This is possible, of course, for many important types of applied forces, including the sinusoidal.

It follows at once from Theorem IV that $x(t)$ satisfies and is determined by the integral equation

$$\frac{1}{p}\frac{F(p)}{H(p)} = \int_0^\infty x(t)e^{-pt}dt. \tag{56}$$

We have thus succeeded, by virtue of Theorem IV in expressing the response of a network to an arbitrary e.m.f. impressed at time $t=o$, by an integral equation of the same form as that expressing the response to a "unit e.m.f." That is to say we have, at least formally, extended the operational calculus explicitly to the case of arbitrary impressed forces.

We now translate the foregoing into the corresponding Operational Theorem.

Theorem VIII

If the operational equation

$$h = 1/H(p)$$

expresses the response of a network to a "unit e.m.f." and if an arbitrary e.m.f. E impressed at time $t=o$, is expressible by the operational equation

$$E = V(p)$$

or the infinite integral

$$\int_0^\infty E(t)e^{-pt}dt = \frac{V(p)}{p}$$

then the response x of the network to the arbitrary force is given by the operational equation

$$x = \frac{V(p)}{H(p)},$$

and $x(t)$ is determined by the integral equation

$$\frac{1}{p}\frac{V(p)}{H(p)} = \int_0^\infty x(t)e^{-pt}dt.$$

Theorem IX

If the operational equation

$$h=1/H(p)$$

is reducible to the form

$$h=\frac{F(p)}{1+\lambda K(p)}$$

where λ is a real parameter, and if the auxiliary functions $f=f(t)$ and $k=k(t)$ are defined by the auxiliary operational equations

$$f=F(p)$$

$$k=K(p)$$

then *$h(t)$ is determined by the Poissan Integral equation*

$$h(t)=f(t)-\lambda\int_0^t h(\tau)k(t-\tau)d\tau.$$

This theorem is of considerable practical importance in connection with the approximate and numerical solution of operational equations when the operational equation and the equivalent Laplace integral equation prove refractory. In such cases, as will be shown later, the numerical solution of the Poissan integral equations can often be rapidly and accurately effected, and in many cases the qualitative properties of $h(t)$ can be deduced from it without detailed numerical solution.

The proof of this theorem proceeds as follows:

By virtue of the relation $h=1/H(p)$ the operational equation

$$h=\frac{F(p)}{1+\lambda K(p)}$$

can be written as

$$h+\lambda\frac{K(p)}{H(p)}=F(p)$$

$$h=F(p)-\lambda\frac{K(p)}{H(p)}.$$

A direct application of Borel's theorem or Theorem IV gives at once the explicit equivalent

$$h(t)=f(t)-\lambda\int_0^t h(\tau)k(t-\tau)d\tau.$$

The preceding theorems, together with the power series and expansion theorem solutions formulate the most important rules of the operational calculus, and are constantly employed in the solution of the electrotechnical problems. On the other hand, the table of infinite integrals furnishes the solution of a set of operational equations, which are of the greatest usefulness in the systematic study of propagation phenomena in transmission systems which will engage our attention. Before taking up this study, however, we shall first solve a few specific problems which will serve as an introduction to asymptotic and divergent solutions involving Heaviside's so-called "fractional differentiation."

Problem A: Current Entering the Non-Inductive Cable

The non-inductive cable is a smooth line with distributed resistance R and capacity C per unit length; for the present we neglect inductance and leakage. A consideration of cable problems leads to some of the most interesting questions relating to operational methods, particularly to questions regarding divergent expansions. It would seem best to allow specific problems to serve as an introduction to these general questions.

The differential equations of the cable are

$$\begin{aligned} RI &= -\frac{\partial}{\partial x}V \\ C\frac{d}{dt}V &= -\frac{\partial}{\partial x}I \end{aligned} \qquad (57)$$

where x is the distance, measured along the cable from any fixed point, I is the current at point x, and V the corresponding potential.

Replacing d/dt by the operator p, we have

$$\begin{aligned} RI &= -\frac{\partial}{\partial x}V \\ pCV &= -\frac{\partial}{\partial x}I. \end{aligned} \qquad (58)$$

Eliminating, successively, V and I from these equations, we get

$$pRCI = \frac{\partial^2}{\partial x^2}I$$

and

$$pRCV = \frac{\partial^2}{\partial x^2}V.$$

These equations have the general solutions

$$V = V_1 e^{-\gamma x} + V_2 e^{\gamma x} \tag{59}$$

$$I = \sqrt{\frac{pC}{R}}\,[V_1 e^{-\gamma x} - V_2 e^{\gamma x}] \tag{60}$$

where

$$\gamma = \sqrt{pRC}. \tag{61}$$

The term in $e^{-\gamma x}$ represents the direct wave and the term in $e^{\gamma x}$ the reflected wave. V_1 and V_2 are constants which must be so chosen as to satisfy the imposed boundary conditions at the terminals of the cable.

For the present we shall assume that the line is infinitely long so that the reflected wave is absent. We shall also assume that a voltage E is impressed directly on the cable at $x = o$: we have then,

$$V = Ee^{-x\sqrt{pCR}} = Ee^{-\sqrt{\alpha p}} \tag{62}$$

$$I = \sqrt{\frac{pC}{R}}\,Ee^{-x\sqrt{pCR}} = \sqrt{\frac{pC}{R}}\,Ee^{-\sqrt{\alpha p}} \tag{63}$$

where α denotes x^2RC.

To convert these to operational equations let us suppose that E is a "unit e.m.f." (zero before, unity after time $t = o$). We have then, in operational notation

$$V = e^{-\sqrt{\alpha p}} \tag{64}$$

$$I = \sqrt{\frac{pC}{R}}\,e^{-\sqrt{\alpha p}}. \tag{65}$$

Now suppose that $x = o$ so that $\alpha = o$, in other words consider a point at the cable terminals. Then

$$V = 1$$

$$I = \sqrt{\frac{pC}{R}}. \tag{66}$$

The first of these equations means that V is simply the impressed voltage, zero before, unity after time $t = o$, as of course, it should be from physical considerations.

Corresponding to the operational equation

$$I = \sqrt{\frac{pC}{R}}. \tag{66}$$

we have the integral equation

$$\sqrt{\frac{C}{R}}\frac{1}{\sqrt{p}}=\int_0^\infty I(t)e^{-pt}dt. \tag{67}$$

The solution of this is known (see formula (c) of the preceding table of integrals): it is

$$I=\sqrt{\frac{C}{R}}\frac{1}{\sqrt{\pi t}}=\sqrt{\frac{C}{\pi Rt}}. \tag{68}$$

Heaviside arrived at this solution from considering the known solution of the same problem in the theory of heat flow. He therefore inferred that the operational equation

$$I=\sqrt{p}$$

has the explicit solution

$$I=1/\sqrt{\pi t}.$$

This is correct; we, however, have derived it directly from the integral equation of the problem and the known integral

$$\frac{1}{\sqrt{p}}=\int_0^\infty e^{-pt}\frac{dt}{\sqrt{\pi t}}. \tag{69}$$

We then see from the foregoing that, if a "unit e.m.f." is impressed on the cable terminals, the current entering the cable is initially infinite and dies away in accordance with the formula $\sqrt{C/\pi Rt}$. The case is, of course, idealized and the infinite initial value of the current results from our ignoring the distributed inductance of the cable, which, no matter how small, keeps the initial current finite, as we shall see later.

Now let us go a step farther; suppose that in addition to distributed resistance R and capacity C, the cable also has distributed leakage G per unit length. The differential equations are now

$$\begin{aligned} RI &= -\frac{\partial}{\partial x}V \\ (Cp+G)V &= -\frac{\partial}{\partial x}I. \end{aligned} \tag{70}$$

Consequently it follows that in the operational equation for the current entering the cable we need only replace Cp by $Cp+G$. Therefore, when leakage is included, equation (66) is to be replaced by

$$I=\sqrt{\frac{pC+G}{R}}=\sqrt{\frac{C}{R}}\sqrt{p+\lambda} \tag{71}$$

where $\lambda=G/C$.

The corresponding integral equation is, of course,

$$\sqrt{\frac{C}{R}}\frac{\sqrt{p+\lambda}}{p}=\int_0^\infty I(t)e^{-pt}dt. \tag{72}$$

We shall give two solutions of this problem; first the solution of the integral equation, and second the typical Heaviside solution directly from the operational equation.

Equation (72) may be written as

$$\sqrt{\frac{C}{R}}\frac{(1+\lambda/p)}{\sqrt{p+\lambda}}=\int_0^\infty I(t)e^{-pt}dt. \tag{73}$$

Now suppose that $J(t)$ is the solution of the equation

$$\frac{1}{\sqrt{p+\lambda}}=\int_0^\infty J(t)e^{-pt}dt \tag{74}$$

it follows at once from Theorems (I) and (II) of the preceding chapter that

$$I(t)=\sqrt{\frac{C}{R}}\left(1+\lambda\int_0^t dt\right)J(t). \tag{75}$$

Also from formula (c) of the table of integrals and Theorem (Va) the solution of (74) is

$$J(t)=\frac{e^{-\lambda t}}{\sqrt{\pi t}} \tag{76}$$

whence

$$I(t)=\sqrt{\frac{C}{\pi R}}\left\{\frac{e^{-\lambda t}}{\sqrt{t}}+\lambda\int_0^t\frac{e^{-\lambda t}}{\sqrt{t}}dt\right\}. \tag{77}$$

The integral appearing in (77) can not be evaluated in finite terms; it is easily expressible as a series, however, by repeated integration by parts. Thus

$$\int^t\frac{e^{-\lambda t}}{\sqrt{t}}dt=2\int_0^t e^{-\lambda t}d\sqrt{t}=2\sqrt{t}\,e^{-\lambda t}+2\lambda\int_0^t e^{-\lambda t}\sqrt{t}\,dt.$$

Proceeding in this way by repeated partial integration we get for the integral term of (77)

$$2\sqrt{t}\,e^{-\lambda t}\left\{1+\frac{2\lambda t}{1.3}+\frac{(2\lambda t)^2}{1.3.5}+\ .\ .\ .\ \right\}. \tag{78}$$

The straightforward Heaviside solution is obtained by expanding the operational equation as follows:

$$I=\sqrt{\frac{C}{R}}\sqrt{p+\lambda}$$

$$=\sqrt{\frac{C}{R}}\left(1+\frac{\lambda}{p}\right)^{1/2}\sqrt{p}$$

$$=\sqrt{\frac{C}{R}}\left[1+\frac{1}{2}\frac{\lambda}{p}-\frac{1}{2.4}\left(\frac{\lambda}{p}\right)^2+\frac{1.3}{2.4.6}\left(\frac{\lambda}{p}\right)^3-\,.\,.\,\right]\sqrt{p}.$$

Identifying $\sqrt{p}$ with $1/\sqrt{\pi t}$ (from known solutions of allied problems) and substituting for $1/p^n$ multiple integrations of the nth order we get

$$I=\sqrt{\frac{C}{\pi Rt}}\left\{1+\frac{(2\lambda t)}{2}-\frac{(2\lambda t)^2}{2.3.4}+\frac{1.3(2\lambda t)^3}{2.3.4.5.6}-\,-\,-\right\}. \tag{79}$$

It can be verified that this solution is convergent and equivalent to (77).

This problem, while simple and of minor technical interest, will serve to introduce us to the very important and interesting question of asymptotic series solutions.

An asymptotic series, for our purposes, may be defined as a series expansion of a function, which, while divergent, may be used for numerical computation, and which exhibits the behavior of the function for sufficiently large values of the argument.

Let us return to equation (77). We observe that the series solution (78) of the definite integral becomes increasingly laborious to compute as the value of t increases. This remark applies with even greater force to the Heaviside solution (79) on account of the alternating character of the series. Right here we have an excellent example of what I regard as Heaviside's exaggerated sense of the importance of series solutions as compared with definite integrals. Consider the solution in the form of (77) as compared with Heaviside's series solution (79). The former is incomparably easier to interpret and to compute, either by numerical integration or by means of an integraph or planimeter. In fact the series (79) is practically unmanageable except for small values of t.

Returning to the question of an asymptotic expansion of the solution (77), we observe that the definite integral appearing in that equation can be written as,

$$\int_0^t \frac{e^{-\lambda t}}{\sqrt{t}}dt=\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt-\int_t^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt \tag{80}$$

provided λ is positive, as it is in this case. Now the value of the infinite integral is known; it is $\sqrt{\pi/\lambda}$. Consequently

$$\int_0^t \frac{e^{-\lambda t}}{\sqrt{t}}dt = \sqrt{\frac{\pi}{\lambda}} - \int_t^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt; \tag{81}$$

furthermore,

$$\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt = -\frac{1}{\lambda}\int_t^\infty \frac{1}{\sqrt{t}}de^{-\lambda t} = \frac{1}{\lambda}\frac{e^{-\lambda t}}{\sqrt{t}} - \frac{1}{2\lambda}\int_0^\infty \frac{e^{-\lambda t}}{t\sqrt{t}}dt.$$

Integrating again by parts we get

$$\frac{1}{\lambda}\frac{e^{-\lambda t}}{\sqrt{t}} - \frac{1}{2\lambda^2}\frac{e^{-\lambda t}}{t\sqrt{t}} + \frac{1.3}{2^2\lambda^2}\int_0^\infty \frac{e^{-\lambda t}}{t^2\sqrt{t}}dt.$$

Continuing this process, we get

$$\begin{aligned}\int_t^\infty \frac{e^{-\lambda t}}{\sqrt{t}}dt = \frac{e^{-\lambda t}}{\lambda\sqrt{t}}\Big[1 - \frac{1}{2\lambda t} + \frac{1.3}{(2\lambda t)^2} - \frac{1.3.5}{(2\lambda t)^3} \\ + \ldots + (-1)^n \frac{1.3.5\ldots(2n-1)}{(2\lambda t)^n}\Big]\end{aligned} \tag{82}$$

$$-\frac{(-1)^n}{\lambda}\frac{1.3.5\ldots(2n+1)}{2(2\lambda)^n}\int_t^\infty \frac{e^{-\lambda t}}{t^{n+1}\sqrt{t}}dt.$$

Now this series is divergent, that is, if we continue out far enough in the series the terms begin to increase in value without limit. On the other hand, if we stop with the nth term the error is represented by the integral term in (82) and this is *less than*

$$\frac{(-1)^n}{\lambda\sqrt{t}}\frac{1.3.5\ldots(2n-1)}{(2\lambda t)^{n-1}}e^{-\lambda t}. \tag{83}$$

Consequently *the error committed in stopping with any term in the series is less than the value of that term.* Therefore if we stop with the smallest term in the series, the error is less than the smallest term and decreases with increasing values of t.

We can therefore write the solution (77) as

$$I \backsim \sqrt{\frac{\lambda C}{R}} + \sqrt{\frac{C}{\pi R t}}e^{-\lambda t}\left\{\frac{1}{2\lambda t} - \frac{1.3}{(2\lambda t)^2} + \frac{1.3.5}{(2\lambda t)^3} - \cdots\right\}. \tag{84}$$

The first term, since $\lambda = G/C$, is simply $\sqrt{G/R}$, the d.c. admittance of the leaky cable. The divergent series shows how the current approaches this final steady value.

In this particular problem no asymptotic solution is derivable directly from the operational equation, at least by the straightforward Heaviside processes. Asymptotic solutions, however, constitute a large and important part of Heaviside's transmission line solutions. We shall therefore discuss next a problem for which Heaviside obtained both convergent and divergent series expansions.

Problem B: Terminal Voltage on Cable with "Unit E.M.F." Impressed on Cable Through Condenser

We now take up a problem for which Heaviside obtained a divergent solution, and which will introduce us to the theory of his divergent solutions and so-called "fractional differentiation." We suppose a "unit e.m.f." impressed on an infinitely long cable of distributed resistance R and capacity C per unit length through a condenser of capacity C_o: required the voltage V at the cable terminals. The operational equation of the problem is derived as follows:—

We know from the problem just discussed that the current entering the cable whose terminal voltage is V, is, in operational notation

$$\sqrt{\frac{Cp}{R}}V.$$

But the current flowing into the condenser is

$$C_o p(1-V)$$

since the voltage across the condenser is $1-V$. Equating these two expressions we get

$$V=\frac{pC_o}{pC_o+\sqrt{pC/R}} \tag{85}$$

which is the operational equation of the problem.

This may be written as

$$V=\frac{1}{1+\frac{1}{C_o}\sqrt{\frac{C}{R}}\frac{1}{\sqrt{p}}} \tag{85}$$

$$=\frac{1}{1+\sqrt{a/p}},$$

where

$$\sqrt{a}=\frac{1}{C_o}\sqrt{C/R}.$$

Now expanding this by the binomial theorem

$$V=1-\sqrt{\frac{a}{p}}+\frac{a}{p}-\frac{a}{p}\sqrt{\frac{a}{p}}+\left(\frac{a}{p}\right)^2-\cdot\cdot$$

$$=1+\frac{a}{p}+\left(\frac{a}{p}\right)^2+\ldots.$$

$$-(1+\frac{a}{p}+\left(\frac{a}{p}\right)^2+\ldots)\sqrt{\frac{a}{p}},$$

$$=1+\frac{at}{1!}+\frac{(at)^2}{2!}+\ldots. \tag{86}$$

$$-\left(\frac{2at}{1}+\frac{(2at)^2}{1.3.}+\frac{(2at)^3}{1.3.5.}+\ldots\right)\frac{1}{\sqrt{\pi at}}$$

by the usual Heaviside rules of "algebrizing."

It is worth while verifying this from the integral equation of the problem. We have

$$\frac{1}{p}\frac{1}{1+\sqrt{a/p}}=\int_0^\infty V(t)e^{-pt}dt. \tag{87}$$

The left hand side can be written as

$$\frac{1}{p-a}-\frac{1}{p-a}\sqrt{\frac{a}{p}}$$

and by the formulas and theorems given in a preceding section the solution can be recognized at once as:—

$$V(t)=e^{at}-\sqrt{\frac{a}{\pi}}e^{at}\int_0^t\frac{e^{-a\tau}}{\sqrt{\tau}}\,d\tau. \tag{88}$$

This can also be written as

$$V(t)=\sqrt{\frac{a}{\pi}}e^{at}\int^\infty\frac{e^{-a\tau}}{\sqrt{\tau}}\,d\tau. \tag{89}$$

If the definite integral of (88) is evaluated by successive partial integrations it will be found in agreement with the Heaviside solution (86).

Now the solution (86) is in powers of t and while absolutely convergent becomes progressively more difficult to interpret and compute as the value of t increases. From (89), however, we can derive a divergent or asymptotic solution applicable both for interpretation and computation, when the value of t is sufficiently large. As

in the example discussed before, the asymptotic expansion results from repeated partial integrations; thus

$$\int_t^\infty \frac{e^{-a\tau}}{\sqrt{\tau}}\,d\tau = -\frac{1}{a}\int_t^\infty \frac{1}{\sqrt{\tau}} d\, e^{-a\tau}$$

$$= \frac{e^{-at}}{a\sqrt{t}} - \frac{1}{2a}\int_t^\infty \frac{e^{-a\tau}}{\tau\sqrt{\tau}}\,d\tau$$

$$= \frac{e^{-at}}{a\sqrt{\tau}} + \frac{1}{2a^2}\int_t^\infty \frac{1}{\tau\sqrt{\tau}} d\, e^{-a\tau}$$

$$= \frac{e^{-at}}{a\sqrt{t}} - \frac{e^{-at}}{2a^2t\sqrt{t}} + \frac{1.3}{2^2a^2}\int_t \frac{e^{-a\tau}}{\tau^2\sqrt{\tau}}\,d\tau$$

and finally

$$\frac{e^{-at}}{a\sqrt{t}}\left\{1 - \frac{1}{2at} + \frac{1.3}{(2at)^2} - \frac{1.3.5}{(2at)^3} + \cdots\right\}. \tag{90}$$

The series (90) is divergent just as is (82) of a preceding problem and the error committed by stopping with the smallest term, is of the same character and subject to the same discussion. With this understanding we write the solution (89) as

$$V(t) \backsim \frac{1}{\sqrt{\pi a t}}\left\{1 - \frac{1}{2at} + \frac{1.3}{(2at)^2} - \frac{1.3.5}{(2at)^3} + \cdots\right\}. \tag{91}$$

For large values of t ($at>5$) this series is accurately and rapidly computable. Furthermore it shows by mere inspection the behavior of $V(t)$ for large values of t, and that it ultimately approaches zero as $1/\sqrt{\pi a t}$.

Let us now see how Heaviside attacked this problem and how he arrived at a divergent solution from the operational formula. Returning to the operational equation (85), it can be written as

$$V = \frac{\sqrt{p/a}}{1+\sqrt{p/a}}. \tag{92}$$

Now expand the denominator by the binomial theorem: we get formally

$$V = \left\{1 - \sqrt{\frac{p}{a}} + \frac{p}{a} - \frac{p}{a}\sqrt{\frac{p}{a}} + \left(\frac{p}{a}\right)^2 - \cdots\right\}\sqrt{\frac{p}{a}}$$

$$= \left(1 + \frac{p}{a} + \left(\frac{p}{a}\right)^2 + \cdots\right)\sqrt{\frac{p}{a}} \tag{93}$$

$$-\left(\frac{p}{a} + \left(\frac{p}{a}\right)^2 + \left(\frac{p}{a}\right)^3 + \cdots\right).$$

Heaviside's procedure at this point was as remarkable as it was successful. He first discarded the second series in integral powers of p as meaningless. He then identified $\sqrt{p}$ with $1/\sqrt{\pi t}$ and replaced p^n by d^n/dt^n in the first series, getting

$$V=\left(1+\frac{1}{a}\frac{d}{dt}+\frac{1}{a^2}\frac{d^2}{dt^2}+\ldots\right)\frac{1}{\sqrt{\pi at}} \tag{94}$$

or, carrying out the indicated differentiation,

$$V=\frac{1}{\sqrt{\pi at}}\left(1-\frac{1}{2at}+\frac{1.3}{(2at)^2}-\frac{1.3.5}{(2at)^3}+\ldots\right)$$

which agrees with (91).

This is a typical example of a Heaviside divergent solution for which he offered no explanation and no proof other than its practical success. His procedure in this respect is quite unsatisfactory and in particular his discarding an entire series without explanation is intellectually repugnant. We shall leave these questions for the present, however; later we shall make a systematic study of his divergent solutions and rationalize them in a satisfactory manner. First, however, we shall take up a specific problem for which Heaviside obtains a divergent solution without discarding any terms.

Problem C: Current Entering a Line of Distributed L, R and C

Consider a transmission line of distributed inductance L, resistance R, and capacity C per unit length. The differential equations of current and voltage are

$$\begin{aligned}\left(L\frac{d}{dt}+R\right)I&=-\frac{\partial}{\partial x}V\\ C\frac{d}{dt}V&=-\frac{\partial}{\partial x}I.\end{aligned} \tag{95}$$

Replacing d/dt by p, we get

$$\begin{aligned}(pL+R)I&=-\frac{\partial}{\partial x}V\\ CpV&=-\frac{\partial}{\partial x}I.\end{aligned} \tag{96}$$

Equations (96) correspond exactly with (58) for the non-inductive cable: except that we must replace R by $pL+R$. For the infinitely

long line, therefore, the operational formula for the current entering the line is

$$I=\sqrt{\frac{pC}{pL+R}}V_o \tag{97}$$

where V_o is the voltage at the line terminals. If this is a "unit e.m.f." we have, as our operational equation,

$$I=\sqrt{\frac{pC}{pL+R}} \tag{98}$$

which can be written as

$$I=\sqrt{\frac{C}{L}}\frac{1}{\sqrt{1+2\lambda/p}} \tag{99}$$

where $\lambda=R/2L$.

The corresponding integral equation is

$$\sqrt{\frac{C}{L}}\frac{1}{\sqrt{p^2+2\lambda p}}=\int_0^\infty e^{-pt}I(t)dt. \tag{100}$$

From either equation (99) or (100) and formula (p) of the table of integrals, we see at once that the solution is

$$I=\sqrt{\frac{C}{L}}e^{-\lambda t}I_o(\lambda t) \tag{101}$$

where $I_o(\lambda t)$ is the Bessel function $J_o(i\lambda t)$, where $i=\sqrt{-1}$. (The function is, however, a pure real.)

Heaviside's procedure, in the absence of any correlation between the operational equation and the infinite integral, was quite different. Remarking, with reference to equation (99), that "the suggestion to employ the binomial theorem is obvious," he expands it in the form

$$I=\sqrt{\frac{C}{L}}\left\{1-\frac{\lambda}{p}+\frac{1.3}{2!}\left(\frac{\lambda}{p}\right)^2-\frac{1.3.5}{3!}\left(\frac{\lambda}{p}\right)^3+\,.\,.\right\} \tag{102}$$

and replaces $1/p^n$ by t^n/n in accordance with the rule discussed in preceding sections. The explicit solution is then

$$I=\sqrt{\frac{C}{L}}\left\{1-\lambda t+\frac{1.3}{(2!)^2}(\lambda t)^2-\frac{1.3.5}{(3!)^2}(\lambda t)^3+\,.\,.\,.\right\} \tag{103}$$

a convergent solution in rising powers of t. As yet, however, he does not recognize this series as the power series expansion of (101), which it is. He does, however, recognize the practical impossibility of using it for computing for large values of t, and remarks "But the binomial theorem furnishes another way of expanding the operator

(operational equation), viz. in rising powers of p." Thus, returning to (99), it can be written as,

$$I=\sqrt{\frac{C}{L}}\frac{\sqrt{p/2\lambda}}{\sqrt{1+p/2\lambda}}. \tag{104}$$

Now expand the denominator by the binomial theorem: we get

$$I=\sqrt{\frac{C}{L}}\left\{1-\frac{p}{4\lambda}+\frac{1.3}{2!}\left(\frac{p}{4\lambda}\right)^2-\frac{1.3.5}{3!}\left(\frac{p}{4\lambda}\right)^3+\ldots\right\}\sqrt{\frac{p}{2\lambda}}. \tag{105}$$

He now identifies $\sqrt{p/2\lambda}$ with $1/\sqrt{2\pi\lambda t}$ and replaces p^n in the series by d^n/dt^n, thus getting finally

$$I=\sqrt{\frac{C}{L}}\frac{1}{\sqrt{2\pi\lambda t}}\left\{1+\frac{1}{8\lambda t}+\frac{(1.3)^2}{2!(8\lambda t)^2}+\frac{(1.3.5)^2}{3!(8\lambda t)^3}+\ldots\right\}. \tag{106}$$

This series solution is divergent: Heaviside recognizes it, however, as the asymptotic expansion of the function $e^{-\lambda t}I_o(\lambda t)$, and thus arrives at the solution

$$I=\sqrt{\frac{C}{L}}e^{-\lambda t}I_o(\lambda t) \tag{101}$$

which we have obtained from our tables of integrals.

Now the divergent expansion (106) is the well known asymptotic expansion of the function $e^{-\lambda t}I_o(\lambda t)$, which is usually derived by difficult and intricate processes. The directness and simplicity with which Heaviside derives it is extraordinary.

We note in this example that no integral powers of p appear in the divergent expansion: consequently no terms are discarded. Otherwise Heaviside's process is as startling and remarkable as in the example discussed in the preceding section.

We shall later encounter many problems in which asymptotic solutions are derivable as in the preceding example. We have sufficient data, however, in these two typical examples to take up a systematic discussion of the theory of Heaviside's divergent solution of the operational equation.

CHAPTER V

The Theory of the Asymptotic Solution of Operational Equations

A study of Heaviside's methods, as exemplified in the preceding examples and in many problems dealt with in his Electromagnetic

Theory, Vol. II, shows that they may be divided into two classes: (I) those of which the operational equation is of the form

$$h = F(p)\sqrt{p} \tag{I}$$

and (II) those of which the operational equation is of the form

$$h = \phi(p^k\sqrt{p}) \tag{II}$$

where k is an integer.

Heaviside himself does not distinguish between the two classes, but employs the following rule for obtaining asymptotic expansion solutions:

If the operational equation

$$h = 1/H(p)$$

can be expanded in the form

$$\begin{aligned} h = {} & a_o + a_1 p + a_2 p^2 + \ldots + a_n p^n + \ldots \\ & (b_o + b_1 p + b_2 p^2 + \ldots + b_n p^n + \ldots)\sqrt{p}, \end{aligned} \tag{107}$$

a solution, usually divergent, is obtained by discarding the first expansion entirely, except for the leading constant terms a_o, replacing $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n in the second expansion, whence an explicit series solution results.

$$h = a_o + \left(b_o + b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2} + \ldots\right)\frac{1}{\sqrt{\pi t}} \tag{108}$$

$$= a_o + \frac{1}{\sqrt{\pi t}}\left(b_o - b_1 \frac{1}{2t} + b_2 \frac{1.3}{(2t)^2} - b_3 \frac{1.3.5}{(2t)^3} + \ldots\right). \tag{109}$$

It should be expressly understood that Heaviside nowhere himself states this rule formally. He does not distinguish between the two cases where integral series in p do and do not appear, although very important mathematical distinctions are involved. Furthermore, in one case he modifies his usual procedure by adding an extra term (Elm. Th. Vol. II, pg. 42–44). It certainly represents, however, his usual procedure in a very large number of problems.

A completely satisfactory theory of the Heaviside Rule, just stated, has not yet been arrived at although we can always verify the divergent solutions in specific problems. Furthermore, it is not as yet known just how general it is, though it certainly works successfully in a large number of physical problems to which it has been applied. Finally we know nothing in general as to the asymptotic character of the resulting expansion. In some cases it leads to an expansion in which the error is less than the last term included, in others re-

markably enough the expansion is everywhere convergent, while in yet others its application leads to a series which is meaningless for a certain range of values of t.

Heaviside himself gives no information which would serve us as a guide in informing us when the rule is applicable and when it is not. Consequently it becomes a matter of practical importance, not only to investigate the underlying mathematical philosophy of the rule and to establish it on the basis of orthodox mathematics, but also to develop if possible a criterion of its applicability. In this investigation we shall have recourse to the integral equation of the problem.

We shall take up first the type of problem (Class I) in which the operational equation is

$$h = \frac{1}{H(p)} = F(p)\sqrt{p} \tag{110}$$

and assume that $F(p)$ admits of the formal power series expansion

$$F(p) = b_o + b_1 p + b_2 p^2 + b_3 p^3 + \dots \tag{111}$$

The corresponding integral equation is

$$\frac{F(p)}{\sqrt{p}} = \int_0^\infty h(t) e^{-pt} dt. \tag{112}$$

We now assume the existence of an auxiliary function $k(t)$, defined and determined by the auxiliary integral equation

$$F(p) = \int_0^\infty k(t) e^{-pt} dt. \tag{113}$$

Now since

$$\frac{1}{\sqrt{p}} = \int_0^\infty e^{-pt} \frac{dt}{\sqrt{\pi t}} \tag{114}$$

it follows from (112), (113), and (114) and Borel's Theorem, or Theorem IV, that

$$h(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{k(\tau)}{\sqrt{t-\tau}} d\tau. \tag{115}$$

Now if we differentiate (113) repeatedly with respect to p and put $p=o$, it follows from the expansion (III) that

$$b_n = (-1)^n \int_0^\infty \frac{t^n}{n!} k(t) dt. \tag{116}$$

This equation presupposes, it should be noted, the convergence of the infinite integrals for all values of n, and therefore imposes severe

restrictions on $k(t)$ and hence on $F(p)$. We shall suppose that these restrictions are satisfied, and discuss them later.

Now (115) can be written as:—

$$h(t)=\frac{1}{\sqrt{\pi t}}\int_0^t d\tau . k(\tau)\,(1-\tau/t)^{-1/2}. \tag{117}$$

It can be shown that, if $k(t)$ satisfies the restrictions underlying (116), the integral (117) has an asymptotic solution obtained as follows:—Expand the factor $(1-\tau/t)^{-1/2}$ by the binomial theorem, replace the upper limit of integration by ∞, and integrate term by term: thus

$$h(t) \sim \frac{1}{\sqrt{\pi t}}\left\{\int_0^\infty k(t)dt+\frac{1}{2t}\int_0^\infty \frac{t}{1!}k(t)dt + \frac{1.3}{(2t)^2}\int_0^\infty \frac{t^2}{2!}k(t)dt + \dots\right\}. \tag{118}$$

Finally from (116) we get

$$h(t) \sim \frac{1}{\sqrt{\pi t}}\left\{b_o - b_1\frac{1}{2t}+b_2\frac{1.3}{(2t)^2}-b_3\frac{1.3.5}{(2t)^3}+\dots\right\} \tag{119}$$

which agrees exactly with the Heaviside rule for this case.

The foregoing says nothing regarding the asymptotic character of the solution. It is easy to see qualitatively, however, that (118) and therefore (119) does represent the behavior of the definite integral (117) for large values of t, provided $k(t)$ converges with sufficient rapidity.

The foregoing analysis may now be summarized in the following proposition:

If the operational equation $h=1/H(p)$ is reducible to the form

$$h=F(p)\sqrt{p}$$

and if $F(p)$ admits of power series expansion in p: thus

$$F(p)=b_o+b_1p+b_2p^2+\dots+b_np^n+\dots.$$

so that, formally,

$$h=(b_o+b_1p+b_2p^2+\dots+b_np^n+\dots)\sqrt{p}$$

an explicit series solution, usually asymptotic, is obtained by replacing $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n(n integral) by d^n/dt^n, whence

$$h(t) \sim \left(b_o+b_1\frac{d}{dt}+b_2\frac{d^2}{dt^2}+\dots\right)\frac{1}{\sqrt{\pi t}}$$

$$\simeq \frac{1}{\sqrt{\pi t}}\left(b_o-b_1\frac{1}{2t}+b_2\frac{1.3}{(2t)^2}-b_3\frac{1.3.5}{(2t)^3}+\dots\right)$$

provided the function $k=k(t)$, defined by the operational equation $k=F(p)$, and the infinite integrals

$$\int_0^\infty t^n k(t)dt \qquad (n=1,2,\ldots.)$$

exist.

We shall now apply the foregoing theory to a physical problem discussed in the last section: namely, the current entering an infinitely long line of inductance L, resistance R and capacity C per unit length. It will be recalled (see equation (100)) that the integral equation of this problem is

$$\sqrt{\frac{C}{L}}\frac{1}{\sqrt{p^2+2\lambda p}}=\int_0^\infty e^{-pt}I(t)dt$$

where $\lambda=R/2L$, and that the solution is

$$I=\sqrt{\frac{C}{L}}e^{-\lambda t}I_o(\lambda t).$$

We can derive the solution in another form appropriate for our purposes by writing

$$\sqrt{\frac{C}{L}}\frac{1}{\sqrt{p}}\frac{1}{\sqrt{p+2\lambda}}=\int_0^\infty e^{-pt}I(t)\,dt$$

Now since

$$\frac{1}{\sqrt{p}}=\int_0^\infty e^{-pt}\frac{dt}{\sqrt{\pi t}}$$

and

$$\frac{1}{\sqrt{p+2\lambda}}=\int_0^\infty e^{-pt}\frac{e^{-2\lambda t}}{\sqrt{\pi t}}dt$$

it follows from Borel's theorem that

$$I=\sqrt{\frac{C}{L}}\frac{1}{\pi}\int_0^t\frac{e^{-2\lambda\tau}}{\sqrt{\tau}\sqrt{t-\tau}}d\tau.$$

Now subject this definite integral (omitting the factor $\sqrt{C/L}$) to the same process applied to (117): we get

$$\frac{1}{\pi\sqrt{t}}\left\{\int_0^\infty\frac{e^{-2\lambda t}}{\sqrt{t}}dt+\frac{1}{2t}\int_0^\infty\frac{\sqrt{t}}{1!}e^{-2\lambda t}\,dt\right.$$
$$\left.+\frac{1.3}{(2t)^2}\int_0^\infty\frac{t\sqrt{t}}{2!}e^{-2\lambda t}dt+\ldots\right\}.$$

The infinite integrals are known and have been evaluated. Substituting their values this series becomes:—

$$\frac{1}{\sqrt{2\pi\lambda t}}\left\{1+\frac{1}{8\lambda t}+\frac{1^2.3^2}{2!(8\lambda t)^2}+\frac{1^2.3^2.5^2}{3!(8\lambda t)^3}+\ldots\right\}$$

which is in fact the well known asymptotic expansion of the function $e^{-\lambda t}I_o(\lambda t)$.

A second example may be worth while. Consider the case of an e.m.f. $e^{-\lambda t}$ impressed at time $t=o$ on a cable of distributed resistance R and capacity C: required the current entering the cable. The required formula is [6]

$$I=\sqrt{\frac{C}{\pi R}}\frac{d}{dt}\int_0^t\frac{e^{-\lambda(t-\tau)}}{\sqrt{\tau}}\,d\tau$$

$$=\sqrt{\frac{C}{\pi R}}\left\{\frac{1}{\sqrt{t}}-\lambda\int_0^t\frac{e^{-\lambda\tau}}{\sqrt{t-\tau}}d\tau\right\} \tag{120}$$

by obvious transformations.

Asymptotic expansion of the definite integral as in the preceding example gives the asymptotic formula

$$I=-\sqrt{\frac{C}{\pi Rt}}\left\{\frac{1}{2\lambda t}+\frac{1.3}{(2\lambda t)^2}+\frac{1.3.5}{(2\lambda t)^3}+\ldots\right\}.$$

The operational formula of the problem is

$$I=\sqrt{\frac{C}{R}}\frac{p}{p+\lambda}\sqrt{p}$$

$$=\sqrt{\frac{C}{R}}\frac{p/\lambda}{1+p/\lambda}\sqrt{p}$$

$$=\sqrt{\frac{C}{R}}\left\{\frac{p}{\lambda}-\left(\frac{p}{\lambda}\right)^2+\left(\frac{p}{\lambda}\right)^3-\ldots\right\}\sqrt{p}.$$

Applying the Heaviside Rule, we get the asymptotic expansion

$$I=-\sqrt{\frac{C}{\pi Rt}}\left\{\frac{1}{2\lambda t}+\frac{1.3}{(2\lambda t)^2}+\frac{1.3.5}{(2\lambda t)^3}+\ldots\right\}$$

which agrees with the preceding formula, derived from the definite integral.

We shall now discuss a specific problem in which the Heaviside Rule breaks down. For example let us take the preceding problem, and

[6] The derivation of the formulas in this problem is left as an exercise for the reader.

replace the applied e.m.f. $e^{-\lambda t}$ by $\sin \omega t$. The formula corresponding to (120) is now

$$I=\omega\sqrt{\frac{C}{\pi R}}\int_0^t \frac{\cos\omega\tau}{\sqrt{t-\tau}}d\tau. \tag{121}$$

If we now attempt to expand the definite integral of (121) in the same way as that of (120), we find that the process breaks down because each component of the infinite integral is now itself infinite. In fact no asymptotic solution of this problem exists.

Let us, however, start with the operational formula: since

$$\int_0^\infty e^{-pt}\sin\omega t.dt=\frac{\omega}{p^2+\omega^2}$$

it is

$$I=\sqrt{\frac{C}{R}}\frac{\omega p}{p^2+\omega^2}\sqrt{p}.$$

Now expand this in accordance with the Heaviside Rule: we get, operationally,

$$I=\sqrt{\frac{C}{R}}\left\{\left(\frac{p}{\omega}\right)-\left(\frac{p}{\omega}\right)^3+\left(\frac{p}{\omega}\right)^5-\dots\right\}\sqrt{p}$$

and explicitly

$$I=-\sqrt{\frac{C}{\pi Rt}}\left\{\frac{1}{2\omega t}-\frac{1.3.5}{(2\omega t)^3}+\dots\right\}$$

which is quite incorrect.[7] The incorrectness of the result will be evident when we remember that the final value of the current is the *steady-state* current in response to $\sin\omega t$, or

$$\sqrt{\frac{\omega C}{2R}}(\cos\omega t+\sin\omega t). \tag{122}$$

This result can be derived directly from (121) by writing it as

$$I=\omega\sqrt{\frac{C}{\pi R}}\left\{\cos\omega t\int_0^t\frac{\cos\omega t}{\sqrt{t}}dt+\sin\omega t\int_0^t\frac{\sin\omega t}{\sqrt{t}}dt\right\}. \tag{123}$$

If the time is made indefinitely great the upper limits of the integrals may be replaced by infinity. The infinite integrals are known: substitution of their known values gives (122).

This example illustrates the care which must be used in applying Heaviside's rules for obtaining divergent solutions and the importance

[7] While this series is incorrect as an asymptotic expansion of the current it has important significance, as we shall see, in connection with the building up of alternating currents.

of having a method of checking the correctness of his processes and results.

We now take up the discussion of the asymptotic expansion solutions of operational equations of the type

$$h = \phi\ (p^k\sqrt{p}) \qquad (k \text{ integral}). \tag{123}$$

In this discussion we shall, as a matter of convenience, assume that $k = o$, so that the equation reduces to the form

$$h = \phi(\sqrt{p}). \tag{123a}$$

This will involve no loss of essential generality, since the analytical theory of the two equations is precisely the same.

The Heaviside Rule for this type of operational equation may be formulated as follows:

If the operational equation $h = 1/H(p)$ is reducible to the form

$$h = \phi\ (p^k\sqrt{p})$$

and if ϕ admits of power series expansion in the argument, thus

$$h = a_o + a_1 p^k\sqrt{p} + a_2 p^{2k+1} + a_3 p^{3k+1}\sqrt{p} + \ldots$$

a series solution, usually divergent and asymptotic, is obtained by discarding integral powers of p, and writing

$$h = a_o + (a_1 p^k + a_3 p^{3k+1} + a_5 p^{5k+2} + \ldots)\sqrt{p}.$$

The explicit series solution then results from replacing $\sqrt{p}$ by $1/\sqrt{\pi t}$, and p^n by d^n/dt^n, whence

$$h \backsim a_o + \left(a_1\frac{d^k}{dt^k} + a_3\frac{d^{3k+1}}{dt^{3k+1}} + a_5\frac{d^{5k+2}}{dt^{5k+2}} + \ldots\right)\frac{1}{\sqrt{\pi t}}$$

$$\backsim a_o + \frac{(-1)^k}{\sqrt{\pi t}}\left(a_1\frac{1.3\ldots(2k-1)}{(2t)^k} - a_3\frac{1.3\ldots(6k+1)}{(2t)^{3k+1}} + \ldots\right).$$

The theory of this series solution will be based on the following proposition, deducible from the identity $\int_0^\infty \frac{e^{-pt}}{\sqrt{\pi t}}dt = 1/\sqrt{p}$.

If the function $F(p)$ of the integral equation

$$F(p) = \int_0^\infty f(t)e^{-pt}dt$$

approaches $1/\sqrt{p}$ as p approaches zero, then $f(t)$ ultimately behaves as $1/\sqrt{\pi t}$: that is, if $F(p) \to 1/\sqrt{p}$ as $p \to o$, then $f(t) \backsim 1/\sqrt{\pi t}$ as $t \to \infty$, provided that $f(t)$ converges to zero, and contains no term or factor which is ultimately oscillatory.

To illustrate what this condition means suppose that

$$f(t)=\frac{a}{\sqrt{\pi t}}+\frac{b\cos\omega t}{\sqrt{\pi t}}$$

then

$$\int_0^\infty f(t)e^{-pt}dt \to a/\sqrt{p} \text{ as } p\to 0,$$

and the oscillatory term in $f(t)$ converges to a higher order. The presence of such oscillatory terms vitiate, therefore, the Heaviside Rule: in the following discussion we shall assume that they are absent.

We are now prepared to discuss the operational equation

$$h=\phi\,(p^k\sqrt{p})$$

and for convenience shall assume that $k=0$ so that the operational equation becomes

$$h=\phi\,(\sqrt{p})$$

of which the corresponding or equivalent integral equation is

$$\frac{1}{p}\phi\,(\sqrt{p})=\int_0^\infty h(t)e^{-pt}dt. \tag{123b}$$

We assume that $\phi(\sqrt{p})$ admits of formal power series expansion in the argument: thus

$$\phi(\sqrt{p})=a_0+a_1\sqrt{p}+a_2p+a_3p\sqrt{p}+a_4p^2+\ \ldots$$

without, however, implying anything regarding the convergence of this expansion.

We now introduce the series of auxiliary functions, $g, g_1, g_2, g_3, \ldots.$ defined by the following scheme

$$\begin{aligned}
g(t)&=h(t)-a_o\\
g_1(t)&=g(t)-\frac{a_1}{\sqrt{\pi t}}\\
g_2(t)&=tg_1(t)+\frac{1}{2}\frac{a_3}{\sqrt{\pi t}}\\
g_3(t)&=t.g_2(t)-\frac{1.3}{2^2}\frac{a_5}{\sqrt{\pi t}}\\
g_4(t)&=t.g_3(t)+\frac{1.3.5}{2^3}\frac{a_7}{\sqrt{\pi t}}\\
&\text{-------------}
\end{aligned} \tag{123c}$$

Successive substitutions in the integral equation (123b) and repeated differentiations with respect to p, lead to the set of formulas,

$$\begin{aligned}
&\int_0^\infty g(t)e^{-pt}dt \sim \frac{a_1}{\sqrt{p}} \text{ as } p \to 0\\
&\int_0^\infty t.g_1(t)e^{-pt}dt \sim \frac{a_3}{2\sqrt{p}} \text{ as } p \to 0\\
&\int_0^\infty t.g_2(t)e^{-pt}dt \sim \frac{1.3}{2^2}\frac{a_5}{\sqrt{p}} \text{ as } p \to 0\\
&\int_0^\infty t.g_3(t)e^{-pt}dt \sim -\frac{1.3.5}{2^3}\frac{a_7}{\sqrt{p}} \text{ as } p \to 0\\
&-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -
\end{aligned} \tag{123d}$$

Now *assuming* that $h(t)$ satisfies the restrictions stated in the preceding proposition, it follows from that proposition, that

$$\begin{aligned}
&g(t) \sim a_1/\sqrt{\pi t} \text{ as } t \to \infty\\
&g_1(t) \sim -\frac{a_3}{2t\sqrt{\pi t}} \text{ as } t \to \infty\\
&g_2(t) \sim \frac{1.3}{2^2 t}\frac{a_5}{\sqrt{\pi t}} \text{ as } t \to \infty\\
&g_3(t) \sim -\frac{1.3.5}{2^3 t}\frac{a_7}{\sqrt{\pi t}} \text{ as } t \to \infty\\
&-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -
\end{aligned} \tag{123e}$$

From the set equations (123d) and (123e) it follows by successive substitutions that

$$h(t) \sim a_o + \frac{1}{\sqrt{\pi t}}\left(a_1 - a_3\frac{1}{2t} + a_5\frac{1.3}{2t^2} - a_7\frac{1.3.5}{(2t)^3} + \dots\right)$$

which agrees with the series gotten by applying the Heaviside Rule.

The defect of this derivation, which, however, appears to be inherent, is that it requires us to know or assume at the outset that $h(t)$ satisfies the required restrictions. Consequently an automatic application of the Heaviside Rule may or may not give correct results. On the other hand if we know that an expansion solution in inverse fractional powers of t exists, the Heaviside Rule gives the series with extraordinary directness and simplicity.

The type of expansion solution just discussed will now be illustrated by some specific problems. The first problem is that of the propagated

voltage in the non-inductive cable in response to a "unit e.m.f". It will be recalled that in a preceding chapter we derived the operational formula

$$V=e^{-\sqrt{\alpha p}} \tag{124}$$

where $\alpha=x^2RC$, for the voltage at distance x from the terminal of a non-inductive cable of distributed resistance R and capacity C, in response to a "unit e.m.f." impressed at point $x=0$. Heaviside's solution of this operational equation proceeds as follows:

Expansion of the exponential function in the usual power series gives

$$V=1-\frac{\sqrt{\alpha p}}{1!}+\frac{\alpha p}{2!}-\frac{\alpha p\sqrt{\alpha p}}{3!}+\frac{(\alpha p)^2}{4!}-\dots$$

which may be rearranged as

$$V=1-\left(1+\frac{\alpha p}{3!}+\frac{(\alpha p)^2}{5!}+\dots\right)\sqrt{\alpha p}+\left(\frac{\alpha p}{2!}+\frac{(\alpha p)^2}{4!}+\frac{(\alpha p)^3}{6!}+\dots\right) \tag{125}$$

Heaviside then discards the series in integral powers of p entirely, replaces $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n in the first series, and then gets

$$\begin{aligned} V&=1-\left(1+\frac{\alpha}{3!}\frac{d}{dt}+\frac{\alpha^2}{5!}\frac{d^2}{dt^2}+\dots\right)\sqrt{\frac{\alpha}{\pi t}} \\ &=1-\sqrt{\frac{\alpha}{\pi t}}\left\{1-\frac{1}{3!}\left(\frac{\alpha}{2t}\right)+\frac{1.3}{5!}\left(\frac{\alpha}{2t}\right)^2-\frac{1.3.5}{7!}\left(\frac{\alpha}{2t}\right)^3+\dots\right\} \end{aligned} \tag{126}$$

or

$$V=1-\sqrt{\frac{\alpha}{\pi t}}\left(1-\frac{1}{3}\left(\frac{\alpha}{4t}\right)+\frac{1}{5.2!}\left(\frac{\alpha}{4t}\right)^2-\frac{1}{7.3!}\left(\frac{\alpha}{4t}\right)^3+\dots\right). \tag{127}$$

This solution is correct, as will be shown subsequently.

A rather remarkable feature of this solution—a point on which Heaviside makes no comment—is that it is absolutely convergent. In other words, a process of expansion which in other problems leads to a divergent or asymptotic solution, here results in a convergent series expansion.

To verify this solution we start with the corresponding integral equation of the problem

$$\frac{1}{p}e^{-\sqrt{\alpha p}}=\int_0^\infty V(t)e^{-pt}dt. \tag{128}$$

It follows from this formula and theorem (V) that

$$V(t)=\int_0^t \phi(t)dt$$

where $\phi(t)$ is determined by the integral equation

$$e^{-\sqrt{\alpha p}} = \int_0^\infty \phi(t) e^{-pt} dt.$$

Now from formula (f) of the table of integrals

$$e^{-\sqrt{\alpha p}} = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}} \int_0^\infty e^{-pt} \frac{e^{-\alpha/4t}}{t\sqrt{t}} dt$$

whence

$$\phi(t) = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}} \frac{e^{-\alpha/4t}}{t\sqrt{t}}$$

and finally

$$V(t) = \frac{1}{\sqrt{\pi}} \int_0^{t'} \frac{e^{-1/\tau}}{\tau\sqrt{\tau}} d\tau, \text{ where } t' = 4t/\alpha. \tag{129}$$

To convert this to the form of (127) we write

$$V(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-1/\tau}}{\tau\sqrt{\tau}} d\tau - \frac{1}{\sqrt{\pi}} \int_{t'}^\infty \frac{e^{-1/\tau}}{\tau\sqrt{\tau}} d\tau. \tag{130}$$

The value of the infinite integral is known to be unity so that

$$V = 1 - \frac{1}{\sqrt{\pi}} \int_{t'}^\infty \frac{e^{-1/\tau}}{\tau\sqrt{\tau}} d\tau. \tag{131}$$

Now in the integral term of (131) expand $e^{-1/\tau}$ in the usual exponential power series and then integrate term by term: the series solution (127) results. This series, while absolutely convergent, is difficult to compute for small values of t; an asymptotic expansion, which can be employed for computation for small values of t is gotten as follows:—

Write (129) as

$$\begin{aligned} V &= \frac{1}{\sqrt{\pi}} \int_0^{t'} \sqrt{\tau}\, de^{-1/\tau} \\ &= \sqrt{\frac{t'}{\pi}} e^{-1/t'} - \frac{1}{2\sqrt{\pi}} \int_0^{t'} \frac{e^{-1/\tau}}{\sqrt{\tau}} d\tau. \end{aligned}$$

Repeated partial integrations of this type lead to the series

$$V = \sqrt{\frac{t'}{\pi}} e^{-1/t'} \left\{ 1 - \left(\frac{t'}{2}\right) + 1.3\left(\frac{t'}{2}\right)^2 - \,.\,.\, \right\}. \tag{132}$$

It is interesting to note, in passing, that an asymptotic solution of this type does not appear to be directly deducible from the operational equation. We observe also that, in this problem, the series in inverse

powers of t is convergent while the series in ascending powers of t is divergent: the converse is the case in the problems discussed previously.

A second specific problem may be stated as follows:

Let a "unit e.m.f." be impressed on an infinitely long non-inductive cable of distributed resistance R and capacity C per unit length through a terminal resistance R_o: required the voltage V on the cable terminals. The formulation of the operational equation of this problem is very simple. It will be recalled that the operational formula for the current entering the cable with terminal voltage V is $V\sqrt{Cp/R}$. But the current is clearly also equal to $(1-V)/R_o$: equating these expressions we get

$$\frac{1-V}{R_o}=V\sqrt{pC/R}$$

whence

$$V=\frac{1}{\sqrt{p/\lambda}+1} \tag{133}$$

where $1/\sqrt{\lambda}=R_o\sqrt{C/R}$. This is the required operational formula.

To derive the Heaviside divergent expansion, expand (133) by the binomial theorem: thus

$$\begin{aligned} V&=1-\sqrt{p/\lambda}\ +(p/\lambda)-(p/\lambda)^{3/2}+\ \ldots \\ &=1-(1+p/\lambda+(p/\lambda)^2+\ \ldots)\sqrt{p/\lambda} \\ &\quad+(p/\lambda+(p/\lambda)^2+(p/\lambda)^3+\ \ldots). \end{aligned} \tag{134}$$

Discard the second series in integral powers of p; replace $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n in the first series, thus getting

$$V=1-\left(1+\frac{1}{\lambda}\frac{d}{dt}+\frac{1}{\lambda^2}\frac{d^2}{dt^2}+\ \ldots\right)\frac{1}{\sqrt{\pi\lambda t}} \tag{135}$$

$$=1-\frac{1}{\sqrt{\pi\lambda t}}\left(1-\frac{1}{2\lambda t}+\frac{1.3}{(2\lambda t)^2}-\ \ldots\right) \tag{136}$$

which is the asymptotic solution of the problem.

To verify this solution we shall consider the more general operational equation

$$h=\frac{1}{p^n\sqrt{p}+1} \qquad (n \text{ integral}) \tag{137}$$

a form of equation to which a number of fairly important problems is reducible. (The parameter λ of equation (133) can be eliminated from explicit consideration by means of theorem VI.)

Multiplying numerator and denominator of equation (137) by $p^n\sqrt{p}-1$, it becomes

$$h=\frac{p^n\sqrt{p}-1}{p^{2n+1}-1}=\frac{p^n}{p^{2n+1}-1}\sqrt{p}-\frac{1}{p^{2n+1}-1} \tag{138}$$

and by direct partial fraction expansion, this is equivalent to

$$h=\frac{\sqrt{p}}{2n+1}\sum_{m=0}^{2n}\frac{p_m^{n+1}}{p-p_m}-\frac{1}{2n+1}\sum_{m=0}^{2n}\frac{p_m}{p-p_m} \tag{139}$$

where

$$p_m=e^{i\frac{2m\pi}{2n+1}} \qquad (m=0,1,2\ldots 2n).$$

Write, for convenience,

$$h=\sum_{m=0}^{2n}h_m$$

and consider the operational equation

$$h_m=\frac{1}{2n+1}\left(\frac{p_m^{n+1}}{p-p_m}\sqrt{p}-\frac{p_m}{p-p_m}\right). \tag{140}$$

By the rules of the operational calculus, fully discussed in preceding chapters, the solution of this is

$$h_m(t)=\frac{1}{2n+1}\left(\frac{p_m^{n+1}}{\sqrt{\pi}}\int_0^t\frac{e^{p_m(t-\tau)}}{\sqrt{\tau}}d\tau+1-e^{p_mt}\right). \tag{141}$$

We have now to distinguish two cases: (1) when the *real part* of p_m is positive, and (2) when the real part is negative.

Taking up case (1) first, the preceding can be written

$$h_m(t)=\frac{1}{2n+1}\left(1+e^{p_mt}\left\{\frac{p_m^{n+1}}{\sqrt{\pi}}\int_0^t\frac{e^{p_m\tau}}{\sqrt{\tau}}d\tau-1\right\}\right) \tag{142}$$

$$=\frac{1}{2n+1}\left(\begin{array}{l}1+e^{p_mt}\left\{\dfrac{p_m^{n+1}}{\sqrt{\pi}}\displaystyle\int_0^\infty\frac{e^{-p_m\tau}}{\sqrt{\tau}}d\tau-1\right\}\\ -\dfrac{p_m^{n+1}}{\sqrt{\pi}}e^{p_mt}\displaystyle\int_t^\infty\frac{e^{-p_mt}}{\sqrt{\tau}}d\tau\end{array}\right) \tag{143}$$

$$=\frac{1}{2n+1}\left(1-\frac{p_m^{n+1}}{\sqrt{\pi}}e^{p_mt}\int_t^\infty\frac{e^{-p_m\tau}}{\sqrt{\tau}}d\tau\right). \tag{144}$$

Repeated integration by parts of the definite integral leads to an asymptotic series, identical with that obtained by applying the Heaviside Rule to the operational equation (137).

If, on the other hand, the *real part* of p_m is negative, we write (141) as

$$h_m(t)=\frac{1}{2n+1}\left(\begin{array}{l}1-e^{p_m t}\\ \quad+\dfrac{p_m^{n+1}}{\sqrt{\pi}}\displaystyle\int_0^t\frac{e^{p_m\tau}}{\sqrt{t-\tau}}d\tau\end{array}\right). \tag{145}$$

The term $e^{p_m t}$ ultimately dies away, and the definite integral can be expanded asymptotically in accordance with the theory discussed under Rule I, again leading to an asymptotic series identical with that given by direct application of the Heaviside Rule to the operational equation.

Consequently since the operational equation in h_n can be asymptotically expanded by means of the Heaviside Rule, the operational equation in $h=\sum h_m$ is similarly asymptotically expandible, and the Heaviside Rule is verified for equation (133).

We have now covered, more or less completely, the theoretical rules and principles of the operational calculus in so far as they can be formulated in general terms. We shall now apply these principles and rules to the solution of important technical problems relating to the propagation of current and voltage along lines. In doing, so, while we shall take advantage of our table of integrals with the corresponding solutions of the operational equation, we shall also sketch Heaviside's own methods of solution.

We shall close this discussion of divergent and asymptotic expansions with a general expansion solution of considerable theoretical and practical importance in the problem of the building-up of alternating currents. It will be recalled from Theorem III that the response of a network of generalized operational impedance $H(p)$ to an e.m.f. $E(t)$ impressed at time $t=o$ is given by the operational formula

$$x=\frac{V(p)}{H(p)}$$

where $E=V(p)$ is the operational equation of the applied e.m.f.: that is, analytically

$$\frac{1}{p}V(p)=\int_0^\infty E(t)e^{-pt}\,dt.$$

Now suppose that the impressed e.m.f. is $\sin \omega t$: then by formula (h) of the table of integrals

$$V(p)=\frac{\omega p}{p^2+\omega^2} \tag{146}$$

and denoting x by x_s

$$x_s = \frac{\omega p}{p^2+\omega^2}\frac{1}{H(p)}. \tag{147}$$

If, on the other hand, the impressed e.m.f. is $\cos \omega t$, then by formula (i)

$$V(p) = \frac{p^2}{p^2+\omega^2} \tag{148}$$

and

$$x = x_c = \frac{p^2}{p^2+\omega^2}\frac{1}{H(p)}. \tag{149}$$

Now let us consider the operational expansion suggested by the Heaviside processes:

$$\begin{aligned} x_s &= \frac{p}{\omega}\left(1+\frac{p^2}{\omega^2}\right)^{-1}\frac{1}{H(p)} \\ &= \left\{\frac{p}{\omega} - \left(\frac{p}{\omega}\right)^3 + \left(\frac{p}{\omega}\right)^5 - \left(\frac{p}{\omega}\right)^7 + \ldots\right\}\frac{1}{(H(p)} \end{aligned} \tag{150}$$

and

$$\begin{aligned} x_c &= \left(\frac{p}{\omega}\right)^2\left(1+\frac{p^2}{\omega^2}\right)^{-1}\frac{1}{H(p)} \\ &= \left\{\left(\frac{p}{\omega}\right)^2 - \left(\frac{p}{\omega}\right)^4 + \left(\frac{p}{\omega}\right)^6 - \left(\frac{p}{\omega}\right)^8 + \ldots\right\}\frac{1}{H(p)}. \end{aligned} \tag{151}$$

Now let us identify $1/H(p)$ with $h(t)$ and replace p^n by d^n/dt^n: we get

$$x_s = \left\{\frac{1}{\omega}\frac{d}{dt} - \frac{1}{\omega^3}\frac{d^3}{dt^3} + \frac{1}{\omega^5}\frac{d^5}{dt^5} - \ldots\right\} h(t) \tag{152}$$

and

$$x_c = \left\{\frac{1}{\omega^2}\frac{d^2}{dt^2} - \frac{1}{\omega^4}\frac{d^4}{dt^4} + \frac{1}{\omega^6}\frac{d^6}{dt^6} - \ldots\right\} h(t). \tag{153}$$

We have now to inquire into the significance of equations (152) and (153), derived from the operational equations of the response of the system of an e.m.f. $\sin \omega t$ and $\cos \omega t$ respectively, impressed at time $t=0$. From the mode of derivation of these expansions from the operational equations it might be inferred that they are the divergent of asymptotic expansions of the operational equations (147) and (149). This would certainly not be an unreasonable inference in the light of the Heaviside expansions we have just been considering. This inference is however, not correct: on the other hand, the series (152) and (153) have a definite physical significance, as we shall now show from the explicit equations of the problem.

By equation (31), the explicit equation for x_s, given operationally by (147), is

$$x_s = \frac{d}{dt}\int_0^t \sin \omega\tau . h(t-\tau)d\tau = \int_0^t \sin \omega(t-\tau)h'(\tau)d\tau + h(o) \sin \omega t \quad (154)$$

where $h'(t) = d/dt \ h(t)$. By a well known trigonometric formula, this is

$$x_s = \sin \omega t \int_0^t \cos \omega t . h'(t)dt - \cos \omega t \int_0^t \sin \omega t . h'(t)dt + h(o) \sin \omega t.$$

Writing

$$\int_0^t dt = \int_0^\infty dt - \int_t^\infty dt$$

this becomes

$$x_s = \sin \omega t \int_0^\infty \cos \omega t . h'(t)dt - \cos \omega t \int_0^\infty \sin \omega t . h'(t)dt$$
$$+ h(o) \sin \omega t - \int_t^\infty \sin \omega(t-\tau)h'(\tau)d\tau. \quad (155)$$

The first three terms are simply the steady-state response to the impressed e.m.f. $\sin \omega t$: that is, they represent the ultimate steady state value of x_s when the transient oscillations have died away. The last term, which we shall denote by T_s, represents the transient oscillations which are set up when the e.m.f. is applied. Thus

$$T_s = -\int_t^\infty \sin \omega(t-\tau)h'(\tau)d\tau. \quad (156)$$

Now from (156)

$$T_s = -\frac{1}{\omega}\int_t^\infty h'(\tau).d.\cos \omega(\tau - t)$$

and integrating by parts

$$T_s = \frac{1}{\omega}\frac{d}{dt}h(t) + \frac{1}{\omega}\int_t^\infty \cos \omega(\tau - t)\frac{d^2}{dt^2}h(\tau)d\tau. \quad (157)$$

Repeating the process of partial integration, we get:

$$T_s = \frac{1}{\omega}\frac{d}{dt}h(t) - \frac{1}{\omega^2}\int_t^\infty \sin \omega(\tau - t)\frac{d^3}{d\tau^3}h(\tau)d\tau. \quad (158)$$

Repeating the process again

$$T_s = \frac{1}{\omega}\frac{d}{dt}h(t) - \frac{1}{\omega^3}\frac{d^3}{dt^3}h(t) + \frac{1}{\omega^4}\int_t^\infty \sin \omega(\tau - t)\frac{d^5}{d\tau^5}h(\tau)d\tau.$$

This process can be repeated indefinitely, and we get

$$T_s=\left(\frac{1}{\omega}\frac{d}{dt}-\frac{1}{\omega^3}\frac{d^3}{dt^3}+\frac{1}{\omega^5}\frac{d^5}{dt^5}-\ldots+\frac{(-1)^{n-1}}{\omega^{2n-1}}\frac{d^{2n-1}}{dt^{2n-1}}\right)h(t)$$
$$+\frac{(-1)^n}{\omega^{2n}}\int_t^\infty \sin\omega(\tau-t)\frac{d^{2n+1}}{dt^{2n+1}}h(\tau)d\tau. \qquad (159)$$

The series expansion (159), except for the remainder term, is identical with the series expansion (152) derived directly from the operational equation. This series may be either convergent or divergent, depending on the frequency $\omega/2\pi$ and the character of the indicial admittance function $h(t)$. In the important problems of the building-up of alternating currents in cables and lines we shall see that, even when divergent, the series is of an asymptotic character and can be employed for computation.

We thus arrive at the following theorem:

If an e.m.f. $\sin\omega t$ is impressed at time $t=0$ on a network or system of generalized indicial admittance $h(t)$, and if the *transient distortion*, T_s, is defined as the instantaneous difference between the actual response of the system and the steady-state response, then T_s can be expressed as the series

$$\left(\frac{1}{\omega}\frac{d}{dt}-\frac{1}{\omega^3}\frac{d^3}{dt^3}+\frac{1}{\omega^5}\frac{d^5}{dt^5}-\ldots+\frac{(-1)^{n-1}}{\omega^{2n-1}}\frac{d^{2n-1}}{dt^{2n-1}}\right)h(t) \qquad (160)$$

with a remainder term

$$\frac{(-1)^n}{\omega^{2n}}\int_t^\infty \sin\omega(\tau-t)\frac{d^{2n+1}}{dt^{2n+1}}h(\tau)d\tau.$$

If the impressed e.m.f. is $\cos\omega t$, the corresponding series for the transient distortion, T_c, is

$$\left(\frac{1}{\omega^2}\frac{d^2}{dt^2}-\frac{1}{\omega_4}\frac{d_4}{dt_4}+\frac{1}{\omega^6}\frac{d^6}{dt^6}-\ldots-\frac{(-1)^n}{\omega^{2n}}\frac{d^{2n}}{dt^{2n}}\right)h(t) \qquad (161)$$

with a remainder term

$$\frac{(-1)^n}{\omega^{2n}}\int_t^\infty \cos\omega(\tau-t)\frac{d^{2n+1}}{dt^{2n+1}}h(\tau)d\tau.$$

The second part of this theorem, relating to the transient distortion, T_c, in response to an e.m.f. $\cos\omega t$, is derived from formula (31) by processes precisely analagous to those employed above in deriving the series expansion for T_s. The derivation will be left to the reader.

To summarize the preceding discussion of the divergent solution of operational equations, it may be said that the theory is as yet rather

unsatisfactory. To the physicist it is unsatisfactory because he requires an automatic rule giving a correct asymptotic expansion by purely algebraic operations without investigations of remainder terms or auxiliary functions. Furthermore, the precise sense in which the expansion asymptotically represents the solution cannot be stated in general, but requires an independent investigation in the case of each individual problem.

On the other hand when an asymptotic expansion is known to exist, the Heaviside Rule finds this expansion with incomparable directness and simplicity, the problem of justifying the expansion being a purely mathematical one, which usually need not trouble the physicist. Furthermore, on the purely mathematical side, the Heaviside Rule is of large interest and should lead to interesting developments in the theory of asymptotic expansions.

(*To be continued*)